International Journal of Research in Engineering, Science and Management
Volume 4, Issue 7, July 2021
https://www.ijresm.com | ISSN (Online): 2581-5792

The Science of Elimination in a Homogeneous Linear System

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***Abstract*: This paper illustrates a comprehensive approach to provide step by step explanation of the structure of a linear system, Cramer's rule and the application of determinant technique towards elimination of the unknown variables in a homogeneous linear system.**

***Keywords*: Linear system, Determinant technique, Cramer's rule, Homogeneous linear equation, Elimination.**

1. Introduction

It is often required to eliminate unknown variables from a system of linear equations. Although it is not a major issue, if number of unknown is two when Classical Algebra is used. However, in this method, significant calculation load is observed when the number of unknown is three or more. So there was a need to develop a shortcut to faster the process of elimination. We are going to use Linear Algebra to achieve the same.

2. Development

For better understanding let's start with a linear system of order 2×2 and then upgrade to a higher order system i.e., 3×3 or more.

System of linear equations with two variables *x* and *y*:

$$a_1x+b_1y=c_1 \ldots\ldots\ldots\ldots(1)$$
$$a_2x+b_2y=c_2 \ldots\ldots\ldots..(2)$$

Using Linear Algebra, (1) and (2) together can be written as

$$\begin{bmatrix} a_1x+b_1y \\ a_2x+b_2y \end{bmatrix}_{2\times1} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_{2\times1}$$

$$\Rightarrow \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}_{2\times2} . \begin{bmatrix} x \\ y \end{bmatrix}_{2\times1} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_{2\times1}$$

$$\Rightarrow A_{2\times2} . X_{2\times1} = B_{2\times1} \ldots\ldots\ldots..(3)$$

Where $A_{2\times2} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}_{2\times2}$, $X_{2\times1} = \begin{bmatrix} x \\ y \end{bmatrix}_{2\times1}$ and $B_{2\times1} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_{2\times1}$ are three matrices.

Note that (3) is a matrix equation. Very interestingly $X_{2\times1}$ is the only unknown matrix. Whereas $A_{2\times2}$ and $B_{2\times1}$ are known matrices. When we try to solve this linear system, in fact, we attempt to solve the matrix $X_{2\times1}$.

The determinant of the matrix $A_{2\times2}$ is given by

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = (a_1b_2 - a_2b_1) \ldots\ldots\ldots..(4)$$

Multiplying (4) by *x*, we get

$$x . \Delta = x . \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1x & b_1 \\ a_2x & b_2 \end{vmatrix} \ldots\ldots\ldots..(5)$$

Further multiplying (5) by *y*, we get

$$x . \Delta . y = y. \begin{vmatrix} a_1x & b_1 \\ a_2x & b_2 \end{vmatrix} = \begin{vmatrix} a_1x & b_1y \\ a_2x & b_2y \end{vmatrix}$$

Applying Column Transformation, we get

$$x . \Delta . y = \begin{vmatrix} a_1x+b_1y & b_1y \\ a_2x+b_2y & b_2y \end{vmatrix} = \begin{vmatrix} c_1 & b_1y \\ c_2 & b_2y \end{vmatrix} = y. \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

$$\Rightarrow x . \Delta = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

Suppose $\Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = (c_1b_2 - c_2b_1) \ldots\ldots\ldots..(6)$

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$\Rightarrow x\,.\,\Delta = \Delta_x$

$\Rightarrow x = \dfrac{\Delta_x}{\Delta}$ ………..(7)

Similarly multiplying (4) by *y*, we get

$$y\,.\,\Delta = y\,.\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 y \\ a_2 & b_2 y \end{vmatrix} \text{ ………..(8)}$$

Further multiplying (8) by *x*, we get

$$y\,.\,\Delta\,.\,x = x.\begin{vmatrix} a_1 & b_1 y \\ a_2 & b_2 y \end{vmatrix} = \begin{vmatrix} a_1 x & b_1 y \\ a_2 x & b_2 y \end{vmatrix}$$

Applying Column Transformation, we get

$$y\,.\,\Delta\,.\,x = \begin{vmatrix} a_1 x & a_1 x + b_1 y \\ a_2 x & a_2 x + b_2 y \end{vmatrix} = \begin{vmatrix} a_1 x & c_1 \\ a_2 x & c_2 \end{vmatrix} = x.\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$\Rightarrow y\,.\,\Delta = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Suppose $\Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = (a_1 c_2 - a_2 c_1)$ ………..(9)

$\Rightarrow y\,.\,\Delta = \Delta_y$

$\Rightarrow y = \dfrac{\Delta_y}{\Delta}$ ………..(10)

Combining (7) and (10), we get

$$x = \frac{\Delta_x}{\Delta},\ y = \frac{\Delta_y}{\Delta} \text{ ………..(11)}$$

Result (11) gives the solution of a 2×2 linear system. It is called *Cramer's Rule*.

Note:

(i) If $c_1 = 0$ and $c_2 = 0$, then the system is called a 2×2 homogeneous system.

(ii) A 2×2 homogeneous system is expressed as

$$a_1 x + b_1 y = 0$$
$$a_2 x + b_2 y = 0$$

(iii) Homogeneous system must have at least one solution. If satisfied by (0,0) only and nothing else, the system is said to have a trivial solution. Whereas if satisfied by infinitely many solutions including (0,0), the system is said to have non-trivial solutions.

(iv) For non-trivial solutions, $\Delta = \Delta_x = \Delta_y = 0$.

(v) If the linear system has infinitely many solutions, that means there exist infinite number of ordered pairs of real numbers (x, y) satisfying the linear system.

(vi) For a homogeneous system,

$$\Delta_x = \begin{vmatrix} 0 & b_1 \\ 0 & b_2 \end{vmatrix} = 0 \text{ and } \Delta_y = \begin{vmatrix} a_1 & 0 \\ a_2 & 0 \end{vmatrix} = 0.$$

So the system will have a trivial solution i.e., satisfied by (0,0) only, if $\Delta \neq 0$. Whereas the solutions are non-trivial, if $\Delta = 0$.

Application:

Eliminate *x* and *y* from the following system.

$$a_1 x + b_1 y = 0$$
$$a_2 x + b_2 y = 0$$

Sol.

It is a 2×2 homogeneous system. Therefore we can eliminate *x* and *y* from the given linear system if and only if there exist solutions other than (0,0). Then the system should have non-trivial solutions. Then $\Delta = 0$.

$$\Rightarrow \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

System of linear equations with three variables *x* , *y* and *z*:

$$a_1 x + b_1 y + c_1 z = d_1 \text{ ………(12)}$$
$$a_2 x + b_2 y + c_2 z = d_2 \text{ ……….(13)}$$
$$a_3 x + b_3 y + c_3 z = d_3 \text{ ………..(14)}$$

Using Linear Algebra, (12), (13) and (14) together can be written as

$$\begin{bmatrix} a_1 x + b_1 y + c_1 z \\ a_2 x + b_2 y + c_2 z \\ a_3 x + b_3 y + c_3 z \end{bmatrix}_{3\times 1} = \begin{bmatrix} d_1 \\ d_2 \\ d_2 \end{bmatrix}_{3\times 1}$$

$$\Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}_{3\times 3} . \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}_{3\times 1}$$

$$\Rightarrow A_{3\times3} . X_{3\times1} = B_{3\times1} \ldots\ldots\ldots..(15)$$

where $A_{3\times3} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}_{3\times3}$, $X_{3\times1} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1}$

and $B_{3\times1} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}_{3\times1}$ are three matrices.

Note that (15) is a matrix equation. Very interestingly $X_{3\times1}$ is the only unknown matrix. Whereas $A_{3\times3}$ and $B_{3\times1}$ are known matrices. When we try to solve this linear system, in fact, you attempt to solve the matrix $X_{3\times1}$.

The determinant of the matrix $A_{3\times3}$ is given by

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \ldots\ldots\ldots..(16)$$

Multiplying (16) by *x*, we get

$$x . \Delta = x . \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix} \ldots\ldots\ldots..(17)$$

Further multiplying (17) by *yz*, we get

$$x . \Delta.yz = yz . \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x & b_1y & c_1z \\ a_2x & b_2y & c_2z \\ a_3x & b_3y & c_3z \end{vmatrix}$$

Applying Column Transformation, we get

$$x . \Delta.yz = \begin{vmatrix} a_1x + b_1y + c_1z & b_1y & c_1z \\ a_2x + b_2y + c_2z & b_2y & c_2z \\ a_3x + b_3y + c_3z & b_3y & c_3z \end{vmatrix} = \begin{vmatrix} d_1 & b_1y & c_1z \\ d_2 & b_2y & c_2z \\ d_3 & b_3y & c_3z \end{vmatrix}$$

$$= yz . \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Rightarrow x . \Delta = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

Suppose $\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$$= d_1(b_2c_3 - b_3c_2) - b_1(d_2c_3 - d_3c_2) + c_1(d_2b_3 - d_3b_2) \ldots\ldots\ldots..(18)$$

$$\Rightarrow x . \Delta = \Delta_x$$

$$\Rightarrow x = \frac{\Delta_x}{\Delta} \ldots\ldots\ldots..(19)$$

Similarly multiplying (16) by *y*, we get

$$y . \Delta = y . \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1y & c_1 \\ a_2 & b_2y & c_2 \\ a_3 & b_3y & c_3 \end{vmatrix} \ldots\ldots\ldots..(20)$$

Further multiplying (20) by *zx*, we get

$$y . \Delta.zx = zx . \begin{vmatrix} a_1 & b_1y & c_1 \\ a_2 & b_2y & c_2 \\ a_3 & b_3y & c_3 \end{vmatrix} = \begin{vmatrix} a_1x & b_1y & c_1z \\ a_2x & b_2y & c_2z \\ a_3x & b_3y & c_3z \end{vmatrix}$$

Applying Column Transformation, we get

$$y . \Delta.zx = \begin{vmatrix} a_1x & a_1x + b_1y + c_1z & c_1z \\ a_2x & a_2x + b_2y + c_2z & c_2z \\ a_3x & a_3x + b_3y + c_3z & c_3z \end{vmatrix} = \begin{vmatrix} a_1x & d_1 & c_1z \\ a_2x & d_2 & c_2z \\ a_3x & d_3 & c_3z \end{vmatrix}$$

$$= zx . \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$\Rightarrow y . \Delta = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

Suppose $\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$

$$= a_1(d_2c_3 - d_3c_2) - d_1(a_2c_3 - a_3c_2) + c_1(a_2d_3 - a_3d_2) \ldots\ldots\ldots..(21)$$

$$\Rightarrow y\,.\,\Delta = \Delta_y$$

$$\Rightarrow y = \frac{\Delta_y}{\Delta} \quad \text{………..(22)}$$

Similarly multiplying (16) by *z*, we get

$$z\,.\,\Delta = z\,.\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 z \\ a_2 & b_2 & c_2 z \\ a_3 & b_3 & c_3 z \end{vmatrix} \quad \text{………..(23)}$$

Further multiplying (23) by *xy*, we get

$$z\,.\,\Delta.xy = xy\,.\begin{vmatrix} a_1 & b_1 & c_1 z \\ a_2 & b_2 & c_2 z \\ a_3 & b_3 & c_3 z \end{vmatrix} = \begin{vmatrix} a_1 x & b_1 y & c_1 z \\ a_2 x & b_2 y & c_2 z \\ a_3 x & b_3 y & c_3 z \end{vmatrix}$$

Applying Column Transformation, we get

$$z\,.\,\Delta.xy = \begin{vmatrix} a_1 x & b_1 y & a_1 x + b_1 y + c_1 z \\ a_2 x & b_2 y & a_2 x + b_2 y + c_2 z \\ a_3 x & b_3 y & a_3 x + b_3 y + c_3 z \end{vmatrix} = \begin{vmatrix} a_1 x & b_1 y & d_1 \\ a_2 x & b_2 y & d_2 \\ a_3 x & b_3 y & d_3 \end{vmatrix}$$

$$= xy\,.\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$\Rightarrow z\,.\,\Delta = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Suppose $\Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

$$= a_1(b_2 d_3 - b_3 d_2) - b_1(a_2 d_3 - a_3 d_2) + d_1(a_2 b_3 - a_3 b_2) \quad \text{………..(24)}$$

$$\Rightarrow z\,.\,\Delta = \Delta_z$$

$$\Rightarrow z = \frac{\Delta_z}{\Delta} \quad \text{………..(25)}$$

Combining (19), (22) and (25), we get

$$x = \frac{\Delta_x}{\Delta},\; y = \frac{\Delta_y}{\Delta},\; z = \frac{\Delta_z}{\Delta} \quad \text{………..(26)}$$

Result (26) gives the solution of a 3×3 linear system. It is called *Cramer's Rule*.

Note:

(i) If $c_1 = 0$, $c_2 = 0$ and $c_3 = 0$, then the system is called a 3×3 homogeneous system.

(ii) A 3×3 homogeneous system is expressed as

$$a_1 x + b_1 y + c_1 z = 0$$
$$a_2 x + b_2 y + c_2 z = 0$$
$$a_3 x + b_3 y + c_3 z = 0$$

(iii) Homogeneous system must have at least one solution. If satisfied by (0,0,0) only and nothing else, the system is said to have a trivial solution. Whereas if satisfied by infinitely many solutions including (0,0,0), the system is said to have non-trivial solutions.

(iv) For non-trivial solutions, $\Delta = \Delta_x = \Delta_y = \Delta_z = 0$.

(v) If the linear system has infinitely many solutions, that means there exist infinite number of ordered triplets of real numbers (*x*, *y*, *z*) satisfying the linear system.

(vi) For a homogeneous system,

$$\Delta_x = \begin{vmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix} = 0, \quad \Delta_y = \begin{vmatrix} a_1 & 0 & c_1 \\ a_2 & 0 & c_2 \\ a_3 & 0 & c_3 \end{vmatrix} = 0$$

and $\Delta_z = \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} = 0$.

So the system will have a trivial solution i.e., satisfied by (0,0,0) only, if $\Delta \neq 0$. Whereas the solutions are non-trivial, if $\Delta = 0$.

Application:

Eliminate *x*, *y* and *z* from the following system.

$$a_1 x + b_1 y + c_1 z = 0$$
$$a_2 x + b_2 y + c_2 z = 0$$
$$a_3 x + b_3 y + c_3 z = 0$$

Sol.

It is a 3×3 homogeneous system. Therefore, we can eliminate *x*, *y* and *z* from the given linear system if and only if there exist solutions other than (0,0,0). Then the system should have non-trivial solutions. Then $\Delta = 0$.

$$\Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 .$$

System of linear equations with *n* number of variables $x_1, x_2, x_3, \ldots\ldots, x_{n-1}$ and x_n :

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots + a_{1n}x_n = \mu_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots + a_{2n}x_n = \mu_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots + a_{3n}x_n = \mu_3$$

....

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots + a_{nn}x_n = \mu_n$$

This system is homogeneous when $\mu_1 = 0$, $\mu_2 = 0$, $\mu_3 = 0$, ………. and $\mu_n = 0$. Like a 2×2 or a 3×3 homogeneous systems, an $n\times n$ homogeneous system will also have a trivial solution i.e., satisfied by (0,0,0,………,0) only and nothing else, if $\Delta \neq 0$. Whereas the solutions are non-trivial, if $\Delta = 0$

; where $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix}$.

Application:

Eliminate $x_1, x_2, x_3, \ldots\ldots, x_{n-1}$ and x_n from the following system.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots + a_{2n}x_n = 0$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots + a_{3n}x_n = 0$$

....

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots + a_{nn}x_n = 0$$

Sol.

It is a $n\times n$ homogeneous system. Therefore we can eliminate $x_1, x_2, x_3, \ldots\ldots, x_{n-1}$ and x_n from the given linear system if and only if there exist solutions other than (0,0,0,….,0). Then the system should have non-trivial solutions. Then $\Delta = 0$.

$$\Rightarrow \begin{vmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix} = 0 \cdot$$

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