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Mapping of a Complex Variable with a Unimodular Quadratic Function where all the Coefficients of the Quadratic are Unity versus Mapping of a Quadratic Function of a Unimodular Complex Variable where all the Coefficients of the Quadratic are Unity

Kalyan Roy*
Life Member, Indian Mathematical Society, Pune, India

***Abstract*: In this paper an attempt has been made to find out the mapping pattern of a complex variable z with a unimodular quadratic function f(z), where all the coefficients of the quadratic are unity. Applying this concept one can conclude the solution set of all possible z or can determine the locus of z on Argand Plane, when f(z) lies on a unit circle whose center is at origin. Also attempted to find out the mapping pattern of a quadratic function f(z) of a unimodular complex variable z, where all the coefficients of the quadratic are unity. Applying this concept one can conclude the solution set of all possible f(z) or can determine the locus of f(z) on Argand Plane, when z lies on a unit circle whose center is at origin.**

***Keywords*: Unimodular complex variable, Unimodular complex function, Complex mapping, Complex Quadratic, Graph based complex analysis**

1. To find out the mapping pattern of a complex variable z ; when f(z)= z² + z + 1 is unimodular, i.e., f (z) lies on a unit circle with centre at origin :

$$|f(z)| = 1$$
$$\Rightarrow |z^2 + z + 1| = 1$$
$$\Rightarrow |(z-\omega)(z-\omega^2)| = 1$$
$$\Rightarrow |z-\omega|\,|z-\omega^2| = 1 \ldots\ldots\ldots(1)$$

; where ω and ω^2 are the imaginary cube roots of unity.

That means the product of the distances of a variable point P(z) from two fixed points A(ω) and B(ω^2) on the Argand Plane is 1.

Suppose $z = x + i\,y$; $x, y \in R$, $i = \sqrt{-1}$.

Now $\omega = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$, $\omega^2 = \frac{-1}{2} - \frac{\sqrt{3}}{2}i$.

So, we apply distance formula and get

$$\sqrt{\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{\sqrt{3}}{2}\right)^2}\cdot\sqrt{\left(x+\frac{1}{2}\right)^2 + \left(y+\frac{\sqrt{3}}{2}\right)^2} = 1$$
$$\Rightarrow \left\{\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{\sqrt{3}}{2}\right)^2\right\}\cdot\left\{\left(x+\frac{1}{2}\right)^2 + \left(y+\frac{\sqrt{3}}{2}\right)^2\right\} = 1$$
$$\Rightarrow \left(x+\frac{1}{2}\right)^4 + 2\left(x+\frac{1}{2}\right)^2\left(y^2+\frac{3}{4}\right) + \left(y^2-\frac{3}{4}\right)^2 = 1$$
$$\Rightarrow \left(x^2+x+\frac{1}{4}\right)^2 + 2\left(x^2+x+\frac{1}{4}\right)\left(y^2+\frac{3}{4}\right) + \left(y^2-\frac{3}{4}\right)^2 = 1$$
$$\Rightarrow (x^2+y^2)^2 + 2x(x^2+y^2) + 3x^2 + 2x - y^2 = 0 \ldots\ldots\ldots(2)$$

Equation (2) is satisfied by infinitely many points (x, y) including (0,0), (-1,0), (0,1), (0,-1), (-1,1), (-1,-1), $\left(\frac{-1}{2}, \frac{\sqrt{7}}{2}\right)$, $\left(\frac{-1}{2}, \frac{-\sqrt{7}}{2}\right)$.

So there are infinitely many solutions of z including 0, -1, i, -i, -1+ i, -1- i, $\frac{-1}{2}+\frac{\sqrt{7}}{2}i$, $\frac{-1}{2}-\frac{\sqrt{7}}{2}i$.

Hence, the solution set of z is
$\left\{z = x + i\,y \;:\; (x^2+y^2)^2 + 2x(x^2+y^2) + 3x^2 + 2x - y^2 = 0\right\}$.

Let's plot the graph (Fig. 1) to understand the mapping pattern.

*Corresponding author: director@kasturieducation.com

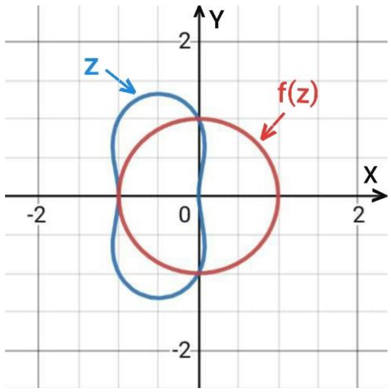

Fig. 1

Note:

(i) Every complex number $f(z)$ on the circle is mapped to another complex number z on the locus traced by z.
(ii) The locus of z is a closed curve.
(iii) The locus of z is symmetric about the line $x=-1/2$.
(iv) The locus of z is symmetric about the line $y=0$.
(v) The locus of z is symmetric about the point $(-1/2, 0)$.

2. To find out the mapping pattern of a quadratic function f(z)= z² + z + 1; when the complex variable z is unimodular, i.e., z lies on a unit circle with centre at origin :

$|z|=1$

Let $z=e^{i\theta}$

$$\Rightarrow f(z)=z^2+z+1=e^{i2\theta}+e^{i\theta}+1$$
$$=(\cos 2\theta+i\sin 2\theta)+(\cos\theta+i\sin\theta)+1$$
$$=(\cos 2\theta+\cos\theta+1)+i(\sin 2\theta+\sin\theta)$$

Suppose $f(z)=x+i\,y$; $x,y\in R$, $i=\sqrt{-1}$.

Then $x+i\,y=(\cos 2\theta+\cos\theta+1)+i(\sin 2\theta+\sin\theta)$

Separating real and imaginary parts, we get

$$x-1=\cos 2\theta+\cos\theta \qquad \ldots\ldots\ldots(3)$$

and

$$y=\sin 2\theta+\sin\theta \qquad \ldots\ldots\ldots(4)$$

Squaring and adding (3) and (4), we get

$$(x-1)^2+y^2=2+2\cos\theta$$

$$\Rightarrow \cos\theta=\frac{(x-1)^2+y^2-2}{2} \qquad \ldots\ldots\ldots(5)$$

Further, (3) can be written as

$$x-1=2\cos^2\theta-1+\cos\theta$$

$$\Rightarrow x-1=2\left\{\frac{(x-1)^2+y^2-2}{2}\right\}^2-1+\left\{\frac{(x-1)^2+y^2-2}{2}\right\}$$

$$\Rightarrow x=\frac{\left\{(x-1)^2+y^2-2\right\}^2}{2}+\frac{(x-1)^2+y^2-2}{2}$$

$$\Rightarrow 2x=(x^2+y^2-2x-1)^2+x^2+y^2-2x-1$$

$$\Rightarrow (x^2+y^2-2x-1)^2+x^2+y^2-4x-1=0$$

$$\Rightarrow (x^2+y^2)^2-4x(x^2+y^2)+3x^2-y^2=0 \qquad \ldots\ldots\ldots(6)$$

Equation (6) is satisfied by infinitely many points (x,y) including (0,0), (1,0), (3,0), (0,1), (0,-1).

So there are infinitely many solutions of $f(z)$ including 0, 1, 3, *i*, -*i*.

Hence, the solution set of $f(z)$ is $\left\{f(z)=x+i\,y\ :\ (x^2+y^2)^2-4x(x^2+y^2)+3x^2-y^2=0\right\}$.

Let's plot the graph (Fig. 2) to understand the mapping pattern.

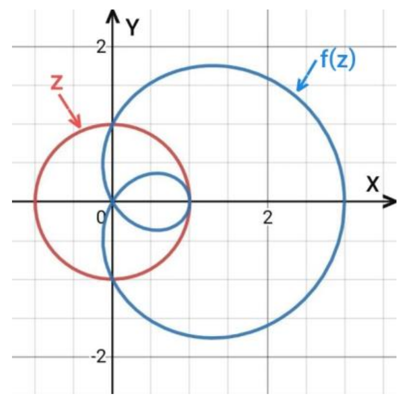

Fig. 2

Note:

(i) Every complex number z on the circle is mapped to another complex number $f(z)$ on the locus traced by $f(z)$.
(ii) The locus of $f(z)$ is a closed curve.
(iii) The locus of $f(z)$ is symmetric about the line $y=0$.
(iv) The locus of $f(z)$ has no point of symmetry.

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