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Operational Calculus of the Sumudu – Finite Mellin Transform

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***Abstract*: The Sumudu–Finite Mellin Transform (SFMT) is a new hybrid integral transform obtained by combining the kernels of the Sumudu transform and the finite Mellin transform. In this paper, the Sumudu-Finite Mellin Transform is introduced and studied in the framework of generalized functions. Suitable testing function spaces are defined using the Gelfand–Shilov technique to establish the distributional form of the transform. Further, important operational properties such as shifting and scaling are obtained. These results extend the theory of hybrid integral transforms and provide a useful analytical tool for solving differential and integral equations over finite domains.**

***Keywords*: Sumudu Transform, Finite Mellin Transform, Sumudu–Finite Mellin Transform, Generalized Functions, Testing Function Spaces, Distribution Theory, Shifting Property, Scaling Property, Operational Calculus.**

1. Introduction

Integral transforms play an important role in solving differential, integral, and integro-differential equations arising in mathematics, physics, and engineering [1], [2]. Among them, the Sumudu transform, introduced by Watugala, has gained importance because of its dimensional preserving property and efficiency in solving initial and boundary value problems [3], [4], [5].

The Sumudu transform has found applications in heat transfer, fluid dynamics, and control theory, making it an effective alternative to the Laplace transform [4], [6]. On the other hand, the Mellin transform is widely used in asymptotic analysis, signal processing, and number theory [7]-[9]. Its finite form, known as the Finite Mellin Transform, extends its applicability to finite interval problems [9], [10].

The theory of generalized transforms has been developed through testing function spaces and distribution theory, mainly using the Gelfand–Shilov framework [11]-[13]. This framework provides a rigorous basis for establishing important properties such as analyticity, inversion, and uniqueness [13], [14].

Recent developments in hybrid integral transforms have shown that combining classical transforms can provide stronger analytical tools for solving differential and integral equations [15]-[17]. Such combined transforms inherit useful operational properties from their constituent transforms and extend their applicability to broader mathematical problems [2], [9].

Motivated by these developments, the Sumudu–Finite Mellin Transform (SFMT) is introduced by combining the kernels of the Sumudu transform and the finite Mellin transform [4], [9], [10]. In this paper, the Sumudu–Finite Mellin Transform is studied in the distributional sense by defining suitable testing function spaces and establishing its shifting, scaling, analyticity, and inversion properties using Gelfand–Shilov techniques [12], [14], [17]. These results extend the theory of hybrid transforms and may be useful for solving differential and integral equations over finite domains [6], [18], [19].

2. Testing Function Spaces

A. The Space $SM^{\beta}_{f,a,b,c,d,\alpha}$:

The Space $SM^{\beta}_{f,a,b,c,d,\alpha}$ is defined as,

$$SM^{\beta}_{f,a,b,c,d,\alpha}=\Big\{\phi:\phi\in E_{+}\,/\,\rho_{a,b,c,d,q,l}\,\phi(t,x)$$

$$\left.=\sup_{\substack{0<t<\infty\\0<x<a}}\left|K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}D_t^l D_x^q\phi(t,x)\right|\le CA^a a^{a\alpha}B^l l^{l\beta}\right\}$$

where, the constants A,B and C depend on the testing function ϕ.

B. The Space $SM_{f,a,b,c,d,\gamma}$:

The Space $SM_{f,a,b,c,d,\gamma}$ is defined as,

$$SM_{f,a,b,c,d,\gamma}=\Big\{\phi:\phi\in E_{+}\,/\,\xi_{a,b,c,d,q,l}\phi(t,x)$$

$$\left.=\sup_{\substack{0<t<\infty\\0<x<a}}\left|K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}D_t^l D_x^q\phi(t,x)\right|\le C_{al}A^q q^{q\gamma}\right\}$$

3. Distributional Generalized Sumudu – Finite Mellin Transform (SM_fT)

For $f(t,x)\in SM^{*\beta}_{f,a,b,c,d,\alpha,}$ where $SM^{*\beta}_{f,a,b,c,d,\alpha}$ is the dual space of $SM^{\beta}_{f,a,b,c,d,\alpha.}$ It contains all distributions of compact support.

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The distributional Sumudu – Finite Mellin Transform is a function of $f(t,x)$ is defined as,

$$SM_f\left\{f(t,x)\right\}=F(u,p)=\left\langle f(t,x),\frac{1}{u}e^{-\frac{t}{u}}\left[\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right]\right\rangle \quad (1)$$

Where, for each fixed $t(0<t<\infty), x(0<x<a), u>0$ and $p>0$, the right-hand side of (3.1) has a sense as an application of $f(t,x)\in SM^{*\beta}_{f,a,b,c,d,\alpha}$ to $\frac{1}{u}e^{-\frac{t}{u}}\left[\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right]\in SM^{\beta}_{f,a,b,c,d,\alpha}.$

4. Operators on the Space $SM^{\beta}_{f,a,b,c,d,\alpha}$

A. Proposition: (Shifting Property for Sumudu Transform)

If $\phi(t,x)\in SM^{\beta}_{f,a,b,c,d,\alpha}$ and ξ is fixed number then $\phi(t+\xi,x)\in SM^{\beta}_{f,a,b,c,d,\alpha}, t+\xi>0$ and $\phi(t+\xi,x)\in S^{v}M^{\beta}_{f,a,b,c,d,\alpha}, t+\xi<0.$

Proof: Consider,

$$\rho_{a,b,c,d,q,l}\,\phi(t+\xi,x)=\underset{I}{Sup}\left|K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}D_t^lD_x^q\phi(t+\xi,x)\right|$$

Put $t+\xi=t'\Rightarrow t=t'-\xi$

$$=\underset{I}{Sup}\left|K_{a,b}(t'-\xi)\lambda_{c,d}(x)x^{q+1}D_{t'}^lD_x^q\phi(t',x)\right|$$

$$\le CA^a a^{a\alpha}B^l l^{l\beta}$$

Thus $\phi(t+\xi,x)\in SM^{\beta}_{f,a,b,c,d,\alpha}$, for $t+\xi>0$.

Similarly, it can be shown that $\phi(t+\xi,x)\in S^{v}M^{\beta}_{f,a,b,c,d,\alpha}$, for $t+\xi<0$.

4.2 Proposition

The Translation (Shifting) operator $\xi:\phi(t,x)\to\phi(t+\xi,x)$ is a topological automorphism on $SM^{\beta}_{f,a,b,c,d,\alpha}$, for $t+\xi>0$, and it is a topological isomorphism from $SM^{\beta}_{f,a,b,c,d,\alpha}$ onto $S^{v}M^{\beta}_{f,a,b,c,d,\alpha}$ for $t+\xi<0$.

4.3 Proposition: (Scaling Property for Sumudu Transform)

If $\phi(t,x)\in SM^{\beta}_{f,a,b,c,d,\alpha}$ and $p>0$, strictly positive number then $\phi(pt,x)\in SM^{\beta}_{f,a,b,c,d,\alpha}$.

Proof: Consider,

$$\rho_{a,b,c,d,q,l}\,\phi(pt,x)=\sup_I\left|K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}D_t^lD_x^q\phi(pt,x)\right|$$

Put $pt=T\Rightarrow t=\frac{T}{p}$

$$=\sup_I\left|K_{a,b}\left(\frac{T}{p}\right)\lambda_{c,d}(x)x^{q+1}D_T^lD_x^q\phi(T,x)\right|$$

$$=M\sup_I\left|K_{a,b}\left(\frac{T}{p}\right)\lambda_{c,d}(x)x^{q+1}D_T^lD_x^q\phi(T,x)\right|$$

Where, M is constant depending on p.

$\le MCA^a a^{a\alpha}B^l l^{l\beta}$

$\le C'A^a a^{a\alpha}B^l l^{l\beta}$, where $C'=MC$

Thus, $\phi(pt,x)\in SM^{\beta}_{f,a,b,c,d,\alpha}$, for $p>0$.

4.4 Proposition

If $\phi(t,x)\in SM^{\beta}_{f,a,b,c,d,\alpha}$ and $p>0$, then the scaling operator $R:SM^{\beta}_{f,a,b,c,d,\alpha}\to SM^{\beta}_{f,a,b,c,d,\alpha}$, defined by $R\phi=\psi$ where $\psi(t,x)=\phi(pt,x)$ is a topological automorphism.

4.5) Proposition: (Shifting Property for Finite Mellin Transform)

If $\phi(t,x)\in SM^{\beta}_{f,a,b,c,d,\alpha}$ and η is any fixed real number then $\phi(t,x+\eta)\in SM^{\beta}_{f,a,b,c,d,\alpha}, x+\eta>0$ and $\phi(t,x+\eta)\in S^{v}M^{\beta}_{f,a,b,c,d,\alpha}, x+\eta<0.$

Proof: Consider,

$$\rho_{a,b,c,d,q,l}\,\phi(t,x+\eta)=\sup_I\left|K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}D_t^lD_x^q\phi(t,x+\eta)\right|$$

Put $x'=x+\eta\Rightarrow x=x'-\eta$

$$=\sup_I\left|K_{a,b}(t)\lambda_{c,d}(x'-\eta)(x'-\eta)^{q+1}D_t^lD_{x'}^q\phi(t,x')\right|$$

$\le CA^a a^{a\alpha}B^l l^{l\beta}$

Thus, $\phi(t,x+\eta)\in SM^{\beta}_{f,a,b,c,d,\alpha}$, for $x+\eta>0$.

Similarly, it can be shown that $\phi(t,x+\eta)\in S^{v}M^{\beta}_{f,a,b,c,d,\alpha}$, for $x+\eta<0$.

4.6) Proposition

The Translation (Shifting) operator $\xi:\phi(t,x)\to\phi(t,x+\eta)$ is a topological automorphism on $SM^{\beta}_{f,a,b,c,d,\alpha}$, for $x+\eta>0$, and it is a topological isomorphism from $SM^{\beta}_{f,a,b,c,d,\alpha}$ onto $S^{v}M^{\beta}_{f,a,b,c,d,\alpha}$ for $x+\eta<0$.

4.7) Proposition: (Scaling Property for Finite Mellin Transform)

If $\phi(t,x)\in SM^{\beta}_{f,a,b,c,d,\alpha}$ and $p>0$, strictly positive number then $\phi(t,px)\in SM^{\beta}_{f,a,b,c,d,\alpha}$.

Proof: Consider,

$$\rho_{a,b,c,d,q,l}\,\phi(t,px)=\sup_I\left|K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}D_t^lD_x^q\phi(t,px)\right|$$

Put $px = X \Rightarrow x = \dfrac{X}{p}$

$$= N \sup_I \left| K_{a,b}(t)\lambda_{c,d}\left(\frac{X}{p}\right) X^{q+1} D_t^l D_X^q \phi(t,X) \right|$$

Where N is constant depending on p.

$\le NCA^a a^{a\alpha} B^l l^{l\beta}$

$\le C'A^a a^{a\alpha} B^l l^{l\beta}$, where $C' = NC$

Thus, $\phi(t,px) \in SM^{\beta}_{f,a,b,c,d,\alpha}$, for $p > 0$.

4.8) Proposition

If $\phi(t,x) \in SM^{\beta}_{f,a,b,c,d,\alpha}$ and $p > 0$, then the scaling operator $R : SM^{\beta}_{f,a,b,c,d,\alpha} \to SM^{\beta}_{f,a,b,c,d,\alpha}$, defined by $R\phi = \psi$ where $\psi(t,x) = \phi(t,px)$ is a topological automorphism.

4.9) Proposition

If $\phi(t,x) \in SM^{\beta}_{f,a,b,c,d,\alpha}$ and ξ and η are fixed real number then $\phi(t+\xi, x+\eta) \in SM^{\beta}_{f,a,b,c,d,\alpha}$,

$t+\xi > 0$ and $x+\eta > 0$ also $\phi(t+\xi, x+\eta) \in S^{\nu}M^{\beta}_{f,a,b,c,d,\alpha}$, $t+\xi < 0$ and $x+\eta < 0$.

Proof: Proof is simple, hence omitted.

4.10) Proposition

If $\phi(t,x) \in SM^{\beta}_{f,a,b,c,d,\alpha}$ and $p > 0$, strictly positive number then $\phi(pt,qx) \in SM^{\beta}_{f,a,b,c,d,\alpha}$.

Proof: Proof is simple, hence omitted.

5. Results on Differential operator $S-$type

5.1) Theorem:

The operator $\phi(t,x) \to D_t\,\phi(t,x)$ is defined on the space $SM^{\beta}_{f,a,b,c,d,\alpha}$ and transforms this space into itself.

Proof: Let $\phi(t,x) \in SM^{\beta}_{f,a,b,c,d,\alpha}$.

If $D_t\,\phi(t,x) = \psi(t,x)$, then we have

$$\begin{aligned}
\rho_{a,b,c,d,q,l}\,\psi(t,x) &= \sup_I \left| K_{a,b}(t)\lambda_{c,d}(x) x^{q+1} D_t^l D_x^q \psi(t,x) \right| \\
&= \sup_I \left| K_{a,b}(t)\lambda_{c,d}(x) x^{q+1} D_t^l D_x^q D_t\,\phi(t,x) \right| \\
&= \sup_I \left| K_{a,b}(t)\lambda_{c,d}(x) x^{q+1} D_t^{l+1} D_x^q\,\phi(t,x) \right| \\
&\le CA^a a^{a\alpha} B^{(l+1)} (l+1)^{(l+1)\beta}
\end{aligned}$$

$\psi(t,x) \in SM^{\beta}_{f,a,b,c,d,\alpha}$ i.e. $D_t\phi(t,x) \in SM^{\beta}_{f,a,b,c,d,\alpha}$.

Or

$$\rho_{a,b,c,d,q,l}\left[D_t\phi(t,x)\right] = \rho_{a,b,c,d,q,l+1}\,\phi(t,x), a,b,c,d,q,l \in \mathbb{N}_0.$$

5.2) Theorem:

The operator $\phi(t,x) \to D_x\,\phi(t,x)$ is defined on the space $SM_{f,a,b,c,d,\gamma}$ and transforms this space into itself.

Proof: Let $\phi(t,x) \in SM_{f,a,b,c,d,\gamma}$.

If $D_x\,\phi(t,x) = \psi(t,x)$, then we have

$$\begin{aligned}
\xi_{a,b,c,d,q,l}\,\psi(t,x) &= \sup_I \left| K_{a,b}(t)\lambda_{c,d}(x) x^{q+1} D_t^l D_x^q \psi(t,x) \right| \\
&= \sup_I \left| K_{a,b}(t)\lambda_{c,d}(x) x^{q+1} D_t^l D_x^q D_x\,\phi(t,x) \right| \\
&= \sup_I \left| K_{a,b}(t)\lambda_{c,d}(x) x^{q+1} D_t^l D_x^{q+1}\,\phi(t,x) \right| \\
&\le C_{al} A^{(q+1)} (q+1)^{(q+1)\gamma}, \ a,q,l = 0,1,2,...
\end{aligned}$$

Therefore, $\psi(t,x) \in SM_{f,a,b,c,d,\gamma}$ i.e. $D_t\phi(t,x) \in SM_{f,a,b,c,d,\gamma}$.

$$\xi_{a,b,c,d,q,l}\left[D_x\phi(t,x)\right] = \xi_{a,b,c,d,q+1,l}\,\phi(t,x).$$

Note that some operators can be defined for the space $SM_{f,a,b,c,d,\alpha}$.

5.3) Proposition

The differential operator of $S-$type $S : \phi(t,x) \to D_t\phi(t,x)$ is a topological automorphism on $SM^{\beta}_{f,a,b,c,d,\alpha}$.

5.4) Proposition

The differential operator of M_f-type $M_f : \phi(t,x) \to D_x\phi(t,x)$ is a topological automorphism on $SM_{f,a,b,c,d,\gamma}$.

6. Results on Differential Operator

6.1) Theorem:

For $m = (m_1, m_2)$ where $m_1, m_2 = 0,1,2,...$, if $\phi(t,x) \in SM^{\beta}_{f,a,b,c,d,\alpha}$ then $\psi(t,x) \in SM^{\beta}_{f,a,b,c,d,\alpha}$ where $\psi(t,x) = D^m\phi(t,x)$. Further the mapping $\Delta = D^m\phi : \phi \to \psi$ is one–one, linear and continuous.

Proof: For $\psi(t,x) \in SM^{\beta}_{f,a,b,c,d,\alpha}$

Consider,

$$\rho_{a,b,c,d,q,l}\,\psi(t,x) = \sup_I \left| K_{a,b}(t)\lambda_{c,d}(x) x^{q+1} D_t^l D_x^q \psi(t,x) \right|$$

$$= \sup_I \left| K_{a,b}(t)\lambda_{c,d}(x) x^{q+1} D_t^l D_x^q D^m\,\phi(t,x) \right|$$

where $m = (m_1, m_2) = (1,1)$.

$$= \sup_I \left| K_{a,b}(t)\lambda_{c,d}(x) x^{q+1} D_t^{l+1} D_x^{q+1}\,\phi(t,x) \right|$$

$$\le CA^a a^{a\alpha} B^l l^{l\beta} \qquad (6.1.1)$$

Thus, $\psi(t,x) \in SM^{\beta}_{f,a,b,c,d,\alpha}$ if $\phi(t,x) \in SM^{\beta}_{f,a,b,c,d,\alpha}$.

It is obviously linear.

It is injective for, if $D^m\phi = 0$ then $\phi = c =$ Constant.

If $c = 0$ then $\phi = 0$ and D is injective. But if $c \ne 0$ then,

$$\sup_I \left| K_{a,b}(t)\lambda_{c,d}(x) x^{q+1} D_t^l D_x^q\, c \right| = \sup_I \left| K_{a,b}(t)\lambda_{c,d}(x) x^{q+1}\, c \right|$$

for $l = q = 0$

As, the right– hand side is not bounded, we conclude that $\phi(t,x) \notin SM^{\beta}_{f,a,b,c,d,\alpha}$.

Which is contradiction.
Hence $'c'$ must be zero and therefore $\phi = 0$.
For continuity we observe from equation (6.1.1) that,
$\rho_{a,b,c,d,q,l}\, D^m \phi(t,x) \le M \rho_{a,b,c,d,q,l}\, \phi(t,x)$
Where M is some constant. Thus, the theorem is proved.

6.2) Theorem:
For $\sigma \in \square$ and $\phi(t,x) \in SM^{\beta}_{f,a,b,c-\sigma,d,\alpha}$,
$E(\phi) = \psi(t,x) = e^{-\sigma x}\phi(t,x) \in SM^{\beta}_{f,a,b,c,d,\alpha}$.
Proof: For $\phi(t,x) \in SM^{\beta}_{f,a,b,c-\sigma,d,\alpha}$,
Consider,

$$
\begin{aligned}
\rho_{a,b,c,d,q,l}\, \psi(t,x) &= \sup_I \left| K_{a,b}(t)\lambda_{c,d}(x) x^{q+1} D_t^l D_x^q \psi(t,x) \right| \\
&= \sup_I \left| K_{a,b}(t)\lambda_{c,d}(x) x^{q+1} D_t^l D_x^q e^{-\sigma x}\, \phi(t,x) \right| \\
&= \sup_I \left| \sum_{i=0}^{1} c_i\, e^{-\sigma x} K_{a,b}(t)\lambda_{c,d}(x) x^{q+1} D_t^l D_x^q\, \phi(t,x) \right| \\
&\le C A^a a^{a\alpha} B^l l^{l\beta}
\end{aligned}
$$

Thus, $\psi(t,x) \in SM^{\beta}_{f,a,b,c,d,\alpha}$ if $\phi(t,x) \in SM^{\beta}_{f,a,b,c-\sigma,d,\alpha}$.
In view of this proposition we have, the exponential multiplier operator.
$E : SM^{\beta}_{f,a,b,c-\sigma,d,\alpha} \to SM^{\beta}_{f,a,b,c,d,\alpha}$ is a topological isomorphism.

6.3) Theorem
For $\tau \in \square$ and $\phi(t,x) \in SM^{\beta}_{f,a-\tau,b,c,d,\alpha}$,
$E(\phi) = \psi(t,x) = e^{-\tau t}\phi(t,x) \in SM^{\beta}_{f,a,b,c,d,\alpha}$.
Proof: Let $\phi(t,x) \in SM^{\beta}_{f,a-\tau,b,c,d,\alpha}$,
Consider,

$$\rho_{a,b,c,d,q,l}\, \psi(t,x) = \sup_I \left| K_{a,b}(t)\lambda_{c,d}(x) x^{q+1} D_t^l D_x^q \psi(t,x) \right|$$

$$= \sup_I \left| K_{a,b}(t)\lambda_{c,d}(x) x^{q+1} D_t^l D_x^q e^{-\tau t} \phi(t,x) \right|$$

$$= \sup_I \left| e^{at} e^{-\tau t} \lambda_{c,d}(x) x^{q+1} D_t^l D_x^q \phi(t,x) \right|$$

$$= \sup_I \left| e^{(a-\tau)t} \lambda_{c,d}(x) x^{q+1} D_t^l D_x^q \phi(t,x) \right| \le C A^a a^{a\alpha} B^l l^{l\beta}$$

Thus $\psi(t,x) \in SM^{\beta}_{f,a,b,c,d,\alpha}$ if $\phi(t,x) \in SM^{\beta}_{f,a-\tau,b,c,d,\alpha}$.
In the view of this proposition, we have the exponential multiplier operator $E : SM^{\beta}_{f,a-\tau,b,c,d,\alpha} \to SM^{\beta}_{f,a,b,c,d,\alpha}$ is a topological isomorphism.

7. Boundedness Theorem

Let $\psi(t,x) \in SM^{*}_{f,a,b,c,d,\alpha}$ and

$$F(u,p) = SM_f\{f(t,x)\} = \left\langle f(t,x), \frac{e^{-t/u}}{u}\left[\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right] \right\rangle,$$

$a < \operatorname{Re} p < b, u > 0$.

Let $\operatorname{supp} f(t,x) \in S_A \cap S_B$ such that
$S_A = \{t : t \in \square^n, |t| \le A, A > 0\}$ and
$S_B = \{x : x \in \square^n, |x| \le B, B > 0\}$, then for each $\varepsilon > 0, \delta > 0$ there exist a constant $C > 0$ and a non−negative integer n such that

$$\left|F(u,p)\right| \le C\left|\frac{1}{u}\right|\left(1+\left|\frac{1}{u}\right|^l\right) e^{\frac{A+\varepsilon}{|u|}} \max_{0\le q\le n}\left[(B+\delta)^{-p} - (B+\delta)^p\right]$$

Proof: Suppose that $\operatorname{supp} f(t,x) \in S_A \cap S_B$ and let $\varepsilon > 0, \delta > 0$.
Choose $\rho \in D(\square^n)$ such that $\rho(t) = 1$ on a neighbourhood of S_A, and $\operatorname{supp} \rho \subset S_{A+\varepsilon}$.
Since $f(t,x) \in SM^{*}_{f,a,b,c,d,\alpha}$ and the boundedness property of generalized functions, there exist a constant C and a non−negative integer n such that

$$F(u,p) = \left| \left\langle f(t,x), \frac{e^{-t/u}}{u}\left[\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right] \right\rangle \right|$$

$$= \left| \left\langle f(t,x), \rho(t)\frac{e^{-t/u}}{u}\left[\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right] \right\rangle \right|$$

$$\le C_1 \max_{\substack{0\le l\le n \\ 0\le q\le n}} \left[\gamma_{a,b,c,d,q,l}\left(\rho(t)\frac{e^{-t/u}}{u}\left[\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right] \right) \right]$$

$$\le C_1 \max_{\substack{0\le l\le n \\ 0\le q\le n}} \sup_{I_1} \left| K_{a,b}(t)\lambda_{c,d}(x) x^{q+1} D_t^l D_x^q \rho(t) \frac{e^{-t/u}}{u}\left[\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right] \right|$$

$$\le C_1 \max_{\substack{0\le l\le n \\ 0\le q\le n}} \sup_{I_1} \left| K_{a,b}(t) \sum_v \binom{l}{v} D_t^{l-v}\rho(t) D_t^v \frac{e^{-t/u}}{u} \lambda_{c,d}(x) x^{q+1} D_x^q \left[\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right] \right|$$

$$\le C_1 \max_{\substack{0\le l\le n \\ 0\le q\le n}} \sup_{I_1} \left| K_{a,b}(t) \sum_v \binom{l}{v} D_t^{l-v}\rho(t) \frac{1}{u}\left(\frac{-1}{u}\right)^v e^{-t/u} \lambda_{c,d}(x) x^{q+1} \right.$$

$$\left. \left\{ a^{2p} P(-p-q) x^{-p-q-1} - P(p-q) x^{p-q-1} \right\} \right|$$

Where $P(-p-q)$ is polynomial in $(-p-q)$ etc.

$$\le C_1 e^{\frac{A+\varepsilon}{|u|}} \max_{\substack{0\le l\le n \\ 0\le q\le n}} \left[\sum_v \binom{l}{v} \left|\frac{1}{u}\right|^{v+1} \right]$$

$$\left| \lambda_{c,d}(x) \left\{ \left(\frac{a}{x}\right)^{2p} P(-p-q) - P(p-q) \right\} x^p \right|$$

$$\le C_1 \left|\frac{1}{u}\right| \left(1 + \left|\frac{1}{u}\right|^l\right) e^{\frac{A+\varepsilon}{|u|}}$$

$$\max_{0\le q\le n} \left| \lambda_{c,d}(x) \left\{ a^{2p} P(-p-q) x^{-p} - x^p P(p-q) \right\} \right|$$

$$\leq C_1 \left|\frac{1}{u}\right| \left(1+\left|\frac{1}{u}\right|^l\right) e^{\frac{A+\varepsilon}{|u|}} \max_{0\leq q\leq n} \left|\lambda_{c,d}(x) a^{2p} P(-p-q) x^{-p}\right|$$

$$-C_1 \left|\frac{1}{u}\right| \left(1+\left|\frac{1}{u}\right|^l\right) e^{\frac{A+\varepsilon}{|u|}} \max_{0\leq q\leq n} \left|\lambda_{c,d}(x) x^{p} P(p-q)\right|$$

$$\leq C_1 \left|\frac{1}{u}\right| \left(1+\left|\frac{1}{u}\right|^l\right) e^{\frac{A+\varepsilon}{|u|}} C' \max_{0\leq q\leq n} \left(B+\delta\right)^{-p}$$

$$-C_1 \left|\frac{1}{u}\right| \left(1+\left|\frac{1}{u}\right|^l\right) e^{\frac{A+\varepsilon}{|u|}} C'' \max_{0\leq q\leq n} \left(B+\delta\right)^{p},$$

Where $C' = \left|\lambda_{c,d}(x) a^{2p} P(-p-q) x^{-p}\right|$ and

$$C'' = \left|\lambda_{c,d}(x) x^{p} P(p-q)\right| \begin{matrix} \leq C \left|\frac{1}{u}\right| \left(1+\left|\frac{1}{u}\right|^l\right) e^{\frac{A+\varepsilon}{|u|}} \max_{0\leq q\leq n} \left(B+\delta\right)^{-p} \\ -C \left|\frac{1}{u}\right| \left(1+\left|\frac{1}{u}\right|^l\right) e^{\frac{A+\varepsilon}{|u|}} \max_{0\leq q\leq n} \left(B+\delta\right)^{p}, \end{matrix}$$

Where, $C = C'C_1$ and $C = C''C_1$.

$$\left|F(u,p)\right| \leq C \left|\frac{1}{u}\right| \left(1+\left|\frac{1}{u}\right|^l\right) e^{\frac{A+\varepsilon}{|u|}} \max_{0\leq q\leq n} \left[\left(B+\delta\right)^{-p} - \left(B+\delta\right)^{p}\right]$$

8. Conclusion

In this paper, the Sumudu–Finite Mellin Transform (SFMT) has been introduced and studied in the distributional sense. Suitable testing function spaces were defined, and important operational properties such as shifting and scaling were established. These results extend the theory of hybrid integral transforms and indicate that SFMT can be a useful tool for solving differential and integral equations over finite domains.

References

[1] L. Debnath and D. Bhatta, *Integral Transforms and Their Applications*, 2nd ed. Boca Raton, FL, USA: Chapman & Hall/CRC, 2007.

[2] I. N. Sneddon, *The Use of Integral Transforms*. New York, NY, USA: McGraw-Hill, 1972.

[3] B. N. Bhosale, *Integral Transform of Generalized Functions*. New Delhi, India: Discovery Publishing House, 2005.

[4] G. K. Watugala, "Sumudu transform: A new integral transform to solve differential equations and control engineering problems," *Math. Eng. Ind.*, vol. 6, no. 4, pp. 319–329, 1993.

[5] F. B. M. Belgacem and A. A. Karaballi, "Sumudu transform fundamental properties investigations and applications," *J. Appl. Math. Stoch. Anal.*, vol. 2006, Art. no. 91083, 2006.

[6] V. K. Gupta, "Applications of Sumudu transform in engineering problems," *Int. J. Math. Sci.*, 2011.

[7] R. B. Paris and D. Kaminski, *Asymptotics and Mellin-Barnes Integrals*. Cambridge, U.K.: Cambridge Univ. Press, 2001.

[8] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*. Oxford, U.K.: Oxford Univ. Press, 1948.

[9] S. M. Khairnar, R. M. Pise, and J. N. Salunke, "Generalized finite Mellin integral transform," *Arch. Appl. Sci. Res.*, vol. 4, no. 2, pp. 1117–1134, 2012.

[10] S. M. Khairnar, R. M. Pise, and J. N. Salunke, "Relation of finite Mellin integral transform with Laplace and Fourier transform," *Contemp. Eng. Sci.*, vol. 4, no. 6, pp. 269–288, 2011.

[11] V. D. Sharma and A. N. Rangari, "Operational calculus on Fourier-finite Mellin transform," *Int. J. Emerg. Innov. Technol.*, vol. 3, no. 4, pp. 161–164, 2013.

[12] V. D. Sharma and A. N. Rangari, "Operation transform formulae of Fourier-Laplace transform," *Int. J. Pure Appl. Sci. Technol.*, vol. 15, no. 2, pp. 62–67, 2013.

[13] A. H. Zemanian, *Distribution Theory and Transform Analysis*. New York, NY, USA: McGraw-Hill, 1965.

[14] A. H. Zemanian, *Generalized Integral Transformations*. New York, NY, USA: Interscience Publishers, 1968.

[15] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, vol. 2. New York, NY, USA: Academic Press, 1968.

[16] V. D. Sharma and A. N. Rangari, "Operational calculus on generalized Fourier-Laplace transform," *Int. J. Sci. Innov. Math. Res.*, vol. 2, no. 11, pp. 862–867, 2014.

[17] A. N. Rangari, R. S. Pawar, and V. D. Sharma, "Testing function spaces and operation transform formulae for Sumudu-Laplace transform," *Int. J. Innov. Res. Technol.*, vol. 11, no. 12, 2025.

[18] V. D. Sharma and A. N. Rangari, "Analyticity of distributional Fourier-Laplace transform," *Int. J. Math. Sci. Eng. Appl.*, 2011.

[19] S. M. Khairnar and R. M. Pise, "Applications of Mellin type integral transform to differential equations," *Int. J. Sci. Res. Publ.*, 2012.