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Fundamental Properties of Absolute-Type Cesàro Sequence Space on Metric Spaces

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***Abstract*: The continuing development and interest of mathematical researchers, both from the purely theoretical and the applied perspective, can still be seen in the field of sequence spaces. One such sequence space is the Cesàro sequence of an absolute space, which J. S. Shiue introduced for real sequences in 1970. Let $1 \le p < \infty$ be real numbers. This space is defined to be the set of all real sequences (x_n) such that $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p < \infty$. In $\mathbb{R}$, the terms $|x_k|$, relates to the standard metric $d_{\mathbb{R}}(a,b) := |a-b|, \forall a,b \in \mathbb{R}$, thus $|x_k| = d_{\mathbb{R}}(x_k, 0)$. Aiming to explore this insight, we construct the definition of absolute Cesàro sequence space on an arbitrary metric space (X,d) with a fixed point $x_0 \in X$, denoted by $ces_p(x_0)$. Then, we define a metric on $ces_p(x_0)$, denoted by ρ_p. Also, we investigate the effect of the choice of $x_0 \in X$ and the fundamental aspects of $(ces_p(x_0), \rho_p)$ such as completeness, the inclusion relation of $ces_p(x_0)$, the ordering of ρ_p on elements of $ces_p(x_0)$ for various p, and the separability of $(ces_p(x_0), \rho_p)$. The primary outcomes indicate that for $1 \le p < \infty$, $(ces_p(x_0), \rho_p)$ is complete if (X,d) is complete; for $1 < p < \infty$, $(ces_p(x_0), \rho_p)$ is separable if and only if (X,d) is separable; and $(ces_p(x_0), \rho_p)$ is not complete if (X,d) is not complete. Also, for $1 \le p < q < \infty$, $ces_p(x_0) \subseteq ces_q(x_0)$. For $p = 1$, we have $ces_1(x_0) = \{(x_0, x_0, \dots)\}$. Hence, $(ces_1(x_0), \rho_1)$ is separable for any metric space (X,d).**

***Keywords*: Metric Space, Cesàro Sequence Space, Absolute type of sequence space, Completeness, Inclusion Relation.**

1. Introduction

Sequence spaces refer to specific spaces in mathematics which consist of sequences with added some mathematical structure. Any sequence can also be considered to be a function defined from the set of natural numbers to a given non-empty set [1]. A classical example of a sequence space is the absolute Cesàro sequence space, ces_p, which consists of all real sequences of the form (x_n) such that that satisfy the condition $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p < \infty$ for all $p \in \mathbb{R}$ with $1 \le p < \infty$.

This space has been a subject of great deal of study. The first comprehensive study was done by Shiue [14], following a problem that was posed by the Dutch Mathematical Society concerning the dual of ces_p, which was also a subject of topological study by Jagers [4] and Leibowitz [9]. Asides from the absolute case, Lee and Ng [12] pioneered the non-absolute case, denoted X_p. For $1 \le p < \infty$, X_p is the space of all real sequences (x_n) that satisfy $\sum_{n=1}^{\infty}\left|\frac{1}{n}\sum_{k=1}^{n}x_k\right|^p < \infty$.

The Cesàro sequence space of absolute type has a rich history, not only as a theory but also in various analysis related fields. In the early works on ces_p, the objects studied were real sequences (x_k) with $x_k \in \mathbb{R}$ for all $k \in \mathbb{N}$. Consider real numbers a, b. Their distance is given by the standard metric on $\mathbb{R}$ defined by $d_{\mathbb{R}}(a,b) = |a-b|$ [6]. Thus, the expression $|x_k|$ in the definition of ces_p can be seen as the distance of x_k from the zero element, that is $|x_k| = d_{\mathbb{R}}(x_k, 0)$. In this regard, Maligranda et al. [10] defined $\|x\|_p$, the Cesàro norm, on ces_p. This norm is given by $\|x\|_p := \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{\frac{1}{p}}$ for all $x = (x_n) \in ces_p$, which Maligranda et al. used to capture other elements of ces_p, that are not present in ℓ^p, namely the B-convex property. For any real number p with $1 < p < \infty$, it is known that ces_p lacks the B-convex property, in contrast to ℓ^p, which is B-convex. Also, Hakim et al. [3] studied ces_p in the same setting as Maligranda et al., that is, real sequences, and showed that ces_p is a Banach space with the Cesàro norm. They also examined various sequence-space characteristics such as solid, BK-space, FK-space, and the AK-property.

Though some other studies take different perspectives on sequence spaces related to Cesàro, they all seem to start from sequences whose terms are real numbers, so the measurement of distance is ultimately expressed through the absolute value on $\mathbb{R}$. For example, Malkowsky et al. [11] studied sequence spaces generated by the Cesàro transformation of real sequences, then through the use of norms and dual norms in the construction of Wulff's crystals, related them to crystallography. In the context of machine learning and data clustering, Khan et al. [5] distance measure proposed an intuitionistic fuzzy distance measure related to a paranormed Cesàro sequence space, and even though it is an applied context, the distance measure is formulated in a numerical form which is, from its essence, based on the absolute value of real numbers.

If we looked at the research above as a whole, we see that it always starts from the case where the terms of the sequence are numbers that belong to a universe which in this case is the set of all real numbers, $\mathbb{R}$. In this universe, we can always calculate the distances between elements using the absolute value

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function defined on $\mathbb{R}$. This case changes, however, when the terms of the sequence are no longer real numbers, or objects that can be represented by real numbers, but instead elements of some general metric space, (X,d), where X is the underlying set, and d is the metric on X, which gives some notion of distance between its elements. In this particular case, the expression $|x_k|$ in the definition of ces_p is not defined and so we need a new definition which is solely dependent on the metric d. Because of the relation $|x_k| = d_{\mathbb{R}}(x_k, 0)$ on $\mathbb{R}$, this generalization intends to substitute $|x_k|$ with the distance to some fixed point $x_0 \in X$, that is, $d(x_k, x_0)$.

Consequently, the purpose of this paper is constructing the definition the absolute type of Cesàro sequence space on a metric space (X,d) for a chosen fixed point $x_0 \in X$, which we will denote by $ces_p(x_0)$. This space is defined for all sequences (x_n) with $x_n \in X$ for all $n \in \mathbb{N}$ such that the series $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(x_k, x_0)\right)^p$ converges for $p \in \mathbb{R}$, $1 \le p < \infty$. This is followed with a discussion on the effect of the choice of the fixed point on $ces_p(x_0)$. Then a function ρ_p is constructed such that $\left(ces_p(x_0), \rho_p\right)$ is a metric space. In addition, the study of certain fundamental properties such as the relations of completeness, inclusion for different values of p to verify that the generalized space produced is of a desirable type with a stable topology. This study extends the classical Cesàro sequence theory and is the first of its type to go beyond $\mathbb{R}$. It also provides a base for further extensions beyond the real numbers.

2. Literature Survey

Some basic concepts and properties used in this discussion are presented as follows.

A. Metric Space

Definition 2.1 [6] Let X be any nonempty set. A function $d: X \times X \to \mathbb{R}$ is called a metric on X if for every $x, y, z \in X$ the following hold:

a. $d(x,y) \ge 0$
b. $d(x,y) = 0$ if and only if $x = y$
c. $d(x,y) = d(y,x)$
d. $d(x,y) \le d(x,z) + d(z,y)$

The set X equipped with d, written (X,d), is called a metric space. The elements of X are called points, and $d(x,y)$ is called the distance from x to y.

Definition 2.2 [1] Let (X,d) be a metric space and let $x_0 \in X$. For a real number $\varepsilon > 0$, the neighborhood of x_0 with radius ε, denoted by $N(x_0;\varepsilon)$, is the set

$$N(x_0;\varepsilon) = \{x \in X : d(x_0, x) < \varepsilon\}$$

Definition 2.3 [15] Let (X,d) be a metric space and $A \subset X$. A point $x \in X$ is called a limit point of A if for every real number $\varepsilon > 0$, there exists a point $x_0 \in N(x;\varepsilon) \cap A$ with $x_0 \ne x$; equivalently, $N(x;\varepsilon) \cap A - \{x\} \ne \emptyset$.

Definition 2.4 [1] Let (x_n) be a sequence in a metric space (X,d). The sequence (x_n) is said to converge to a point $x \in X$ if for every real number $\varepsilon > 0$, there exists a natural number K such that for every $n \in \mathbb{N}$ with $n \ge K$, we have $x_n \in N(x;\varepsilon)$.

Definition 2.5 [1] Let (X,d) be a metric space. A sequence (x_n) in X is called a Cauchy sequence if for every real number $\varepsilon > 0$, there exists a natural number H such that for all natural numbers m, n with $m \ge H$ and $n \ge H$, we have $d(x_n, x_m) < \varepsilon$.

Definition 2.6 [1] A metric space (X,d) is called complete if every Cauchy sequence in X converges to some element $x \in X$.

B. Infinite Series

Definition 2.7 [1] Let (x_n) be a sequence of real numbers. The infinite series generated by (x_n) is a new sequence (s_k) defined by $s_k = \sum_{n=1}^{k} x_n$. Furthermore, the infinite series is written as $\sum_{n=1}^{\infty} x_n$, where x_n are called the terms of the series and s_k is called the k-th partial sum of the series

Definition 2.8 [1] The series $\sum_{n=1}^{\infty} x_n$ is said to converge if the sequence of partial sums (s_k) converges, that is, if there exists $S \in \mathbb{R}$ such that $\lim_{k\to\infty} s_k = S$. In this case, S is called the sum (value) of the series and it is written $\sum_{n=1}^{\infty} x_n = S$.

Theorem 2.9 [1] If the series $\sum_{n=1}^{\infty} x_n$ converges, then we have $\lim(x_n) = 0$

Theorem 2.10 [1] If $p > 1$, then the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges

C. Mikowski Inequality

Theorem 2.11 [2] Given real sequences (x_n) and (y_n). If $1 \le p < \infty$, then

$$\left(\sum_{n=1}^{\infty}|x_n + y_n|^p\right)^{\frac{1}{p}} \le \left(\sum_{n=1}^{\infty}|x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}|y_n|^p\right)^{\frac{1}{p}}$$

D. Cesàro Sequence Space of Absolute Type

Definition 2.12 [8] The Cesàro sequence space of absolute type, denoted by ces_p, is the set of all real sequences (x_n) satisfying $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p < \infty$ for real number p with $1 \le p < \infty$. Symbolically,

$$ces_p = \left\{(x_n): \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p < \infty\right\}$$

To understand this definition, several examples are given below.

Example 2.13 The real sequence $X = \left(\frac{1}{n} - \frac{1}{n+1}\right)$ is an element of ces_p for every real number p with $1 < p < \infty$.

Proof: Given any natural number n. For each $k = 1,2,\dots,n$ we have $\frac{1}{k} > \frac{1}{k+1}$, hence $\left|\frac{1}{k} - \frac{1}{k+1}\right| = \frac{1}{k} - \frac{1}{k+1}$. Thus, for every $n \in \mathbb{N}$, $\frac{1}{n}\sum_{k=1}^{n}|x_k| = \frac{1}{n}\sum_{k=1}^{n}\left|\frac{1}{k} - \frac{1}{k+1}\right| = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right)$

$$= \frac{1}{n}\left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \frac{1}{n}\left(1 - \frac{1}{n+1}\right)$$

$$= \frac{1}{n}\left(\frac{n}{n+1}\right)$$

$$= \frac{1}{n+1}$$

Since $\frac{1}{n+1} \le \frac{1}{n}$ for every $n \in \mathbb{N}$, we obtain $\frac{1}{n}\sum_{k=1}^{n}|x_k| \le \frac{1}{n}$, $\forall n \in \mathbb{N}$. Therefore, for any real number p with $1 < p < \infty$, $\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p \le \left(\frac{1}{n}\right)^p = \frac{1}{n^p}$. Summing over n, we get $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p \le \sum_{n=1}^{\infty}\frac{1}{n^p}$. By Theorem 2.13, the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges for every $p > 1$. Hence, we obtain $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p < \infty$. Thus, by definition, $X \in ces_p$ for every real number p with $1 < p < \infty$.

Example 2.14 The constant sequence $X = (c, c, c, \dots)$ for some real number $c > 0$ is not an element of ces_p for every real number p with $1 \le p < \infty$.

Proof: Given any natural number n. Since $X = (c, c, c, \dots)$, we have $x_k = c$ for each $k = 1,2,\dots,n$. Hence, for every $n \in \mathbb{N}$, $\frac{1}{n}\sum_{k=1}^{n}|x_k| = \frac{1}{n}\sum_{k=1}^{n}|c| = \frac{1}{n}n|c| = |c| = c$, so $\frac{1}{n}\sum_{k=1}^{n}|x_k| = c, \forall n \in \mathbb{N}$. Let p be any real number with $1 \le p < \infty$. We have $\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p = c^p$. Summing over $n \in \mathbb{N}$, we obtain $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p = \sum_{n=1}^{\infty} c^p$. Consider the sequence of partial sums of the infinite series, namely (s_n) with $s_n = \sum_{k=1}^{n} c^p = nc^p$. Then (s_n) is unbounded, hence (s_n) diverges. So, the series $\sum_{n=1}^{\infty} c^p$ diverges. Hence, $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p = \sum_{n=1}^{\infty} c^p$ diverges. Therefore, the constant sequence $X = (c, c, c, \dots) \notin ces_p$ for every real number p with $1 \le p < \infty$.

3. Main Results

A. Cesàro Sequence Space of Absolute Type on A Metric Space

In this section, (X, d) represents a metric space, with X being the set and d being the metric. The classical absolute-type Cesàro sequence space on $\mathbb{R}$ with the standard metric was stated in Definition 2.12. The idea is here generalized to an arbitrary metric space by substituting $|x_k|$ with $d(x_k, x_0)$ for some fixed reference point $x_0 \in X$.

Definition 3.1 Let (X, d) be a metric space, $p \in \mathbb{R}$ where $1 \le p < \infty$, and $x_0 \in X$. The absolute-type Cesàro sequence space on (X, d) with reference point x_0, denoted by $ces_p(x_0)$, is the collection of all sequences (x_n) with $x_n \in X$ for every $n \in \mathbb{N}$ such that

$$ces_p(x_0) \coloneqq \left\{(x_n) : \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(x_k, x_0)\right)^p < \infty\right\}$$

Some examples will be provided. Keep in mind that if $X = \mathbb{R}$, d is the standard metric, and $x_0 = 0$, $ces_p(x_0)$ coincides with the classical ces_p in Definition 2.12. Therefore, the examples concentrate on other types of metric spaces.

Examples 3.2 Let (X, d) be a metric space with $X = \mathbb{R}$ and the standard metric $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Fix a reference point $x_0 = 1 \in \mathbb{R}$. Then the absolute-type of Cesàro sequence space over $(\mathbb{R}, d)$ with reference point 1 is the collection of all real sequences (x_n) satisfying

$$ces_p(1) = \left\{(x_n) : \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k - 1|\right)^p < \infty\right\}$$

Where p is the real number statisfy $1 \le p < \infty$.

After providing the definition of $ces_p(1)$, we give some examples of real sequences, in particular, those that are in $ces_p(1)$ and those that are not.

a. The constant real sequence $(c, c, c, \dots)$ belongs to $ces_p(1)$ for every real number p with $1 \le p < \infty$ if and only if $c = 1$.

Proof: Let $(c, c, c, \dots)$ be a constant sequence with $c \in \mathbb{R}$. Given any natural number n. For each $k = 1,2,\dots,n$, we have $x_k = c$. Hence for all $n \in \mathbb{N}$, $\frac{1}{n}\sum_{k=1}^{n}|x_k - 1| = \frac{1}{n}\sum_{k=1}^{n}|c - 1| = \frac{1}{n}n|c - 1| = |c - 1|$. Therefore, for every $p \in \mathbb{R}$ with $1 \le p < \infty$, $\left(\frac{1}{n}\sum_{k=1}^{n}|x_k - 1|\right)^p = |c - 1|^p, \forall n \in \mathbb{N}$.

Since $c \in \mathbb{R}$, we consider the following cases

Case $c \ne 1$: Then $|c - 1| > 0$, and consequently $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k - 1|\right)^p = \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|c - 1|\right)^p = \sum_{n=1}^{\infty}|c - 1|^p$ diverges. Thus, when $c \ne 1$, the sequence $(c, c, c, \dots) \notin ces_p(1)$

Case $c = 1$: Then |c-1|=0, so $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k - 1|\right)^p = \sum_{n=1}^{\infty} 0 = 0 < \infty$. Hence, for $c = 1$, the sequence $(c, c, c, \dots) = (1,1,1,\dots)$ belongs to $ces_p(1)$ for every real number p with $1 \le p < \infty$.

b. The real sequence (a_n) where $a_n = \begin{cases} 0, n\ odd \\ 2, n\ even \end{cases}$ is not an element of $ces_p(1)$ for every real number p with $1 \le p < \infty$.

Proof: Given any natural number n. Notice that for each $k = 1,2,\dots,n$,

$$|a_k - 1| = \begin{cases} |0 - 1| = 1, k\ odd \\ |2 - 1| = 1, k\ even \end{cases}$$

Hence $|a_k - 1| = 1, \forall k = 1,2,\dots,n$. Therefore, for every $n \in \mathbb{N}, \frac{1}{n}\sum_{k=1}^{n}|a_k - 1| = \frac{1}{n}\sum_{k=1}^{n} 1 = \frac{1}{n}n = 1$. Let p be any real number with $1 \le p < \infty$, then we have $\left(\frac{1}{n}\sum_{k=1}^{n}|a_k - 1|\right)^p = 1^p = 1$. Summing over n yields $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|a_k - 1|\right)^p = \sum_{n=1}^{\infty} 1 = \infty$. Therefore $(a_n) \notin ces_p(1)$ for every real number p with $1 \le p < \infty$.

We now consider $\mathbb{R}$ with $d(x,y) = \min\{1, |x-y|\}$ and a reference point $x_0 = 2$; the corresponding space is provided in this example below.

Example 3.3 Let (X,d) be a metric space with $X = \mathbb{R}$ and $d(x,y) = \min\{1, |x-y|\}, \forall x,y \in \mathbb{R}$. Fix a reference point $x_0 = 2 \in \mathbb{R}$. Then the absolute-type of Cesàro sequence space over $(\mathbb{R}, d)$ with reference point $x_0 = 2$ is the collection of all real sequences (x_n) satisfying

$$ces_p(2) = \left\{(x_n): \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\min\{1, |x_k - 2|\}\right)^p < \infty\right\}$$

Where p is the real number statisfy $1 \le p < \infty$.

We will now discuss examples of both real sequences that belong to this space and those which do not

a. The real sequence $\left(2 + \frac{1}{2^n}\right)$ belongs to $ces_p(2)$ for every real number p with $1 < p < \infty$.

Proof: Let $n \in \mathbb{N}$ be arbitrary. For each $k = 1,2,\dots,n$ notice that $\min\{1, |x_k - 2|\} = \min\left\{1, \left|2 + \frac{1}{2^k} - 2\right|\right\} = \min\left\{1, \left|\frac{1}{2^k}\right|\right\} = \min\left\{1, \frac{1}{2^k}\right\}$.

Since $2^k \ge 2, \forall k = 1,2,\dots,n$, we have $\frac{1}{2^k} \le \frac{1}{2} < 1$. Hence $\min\left\{1, \frac{1}{2^k}\right\} = \frac{1}{2^k}, \forall k = 1,2,\dots,n$. we know that the series $\sum_{k=1}^{\infty}\frac{1}{2^k}$ is a geometric series which converges to 1.

Thus, for every $n \in \mathbb{N}$, $0 \le \sum_{k=1}^{n}\frac{1}{2^k} \le \sum_{k=1}^{\infty}\frac{1}{2^k} = 1$. Consequently, $0 \le \frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^k} \le \frac{1}{n}, \forall n \in \mathbb{N}$. Now let p be any real number with $1 \le p < \infty$, then we have $\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^k}\right)^p \le \frac{1}{n^p}$. Summing both sides over n yields $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^k}\right)^p \le \sum_{n=1}^{\infty}\frac{1}{n^p}$. By Theorem 2.2, the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges for every real number $p > 1$. Hence, $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^k}\right)^p \le \sum_{n=1}^{\infty}\frac{1}{n^p} < \infty$.

Since,

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^k}\right)^p = \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\min\{1, |x_k - 2|\}\right)^p$$

It follows that $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\min\{1, |x_k - 2|\}\right)^p < \infty$. Therefore, the sequence $\left(2 + \frac{1}{2^n}\right)$ is an element of $ces_p(2)$ for every real number p with $1 < p < \infty$.

b. The real sequence (x_n) defined by $x_n = \begin{cases} 3, n\ even \\ n^2, n\ odd \end{cases}$ is not an element of $ces_p(2)$ for every real number p with $1 \le p < \infty$.

Proof: Let $n \in \mathbb{N}$ be arbitrary. If n is even, then $x_n = 3$, hence $d(x_n, 2) = \min\{1, |3-2|\} = \min\{1,1\} = 1$. If n is odd, then $x_n = n^2$, so $d(x_n, 2) = \min\{1, |n^2 - 2|\}$. We consider odd n in the following cases.

Case $n = 1$:

$$d(x_n, 2) = \min\{1, |1^2 - 2|\} = \min\{1, 1\} = 1$$

Case $n \ge 3$:

Since $n^2 \ge 9$, we have $n^2 - 2 \ge 7$, hence $|n^2 - 2| \ge 7 > 1$. Therefore, $d(x_n, 2) = \min\{1, |n^2 - 2|\} = 1$.

From both cases, for every odd n we obtain $d(x_n, 2) = 1$. Together with the even case, it follows that $d(x_n, 2) = 1, \forall n \in \mathbb{N}$. Hence, for every $n \in \mathbb{N}$,

$$\frac{1}{n}\sum_{k=1}^{n} d(x_k, 2) = \frac{1}{n}\sum_{k=1}^{n}\min\{1, |x_k - 2|\} = \frac{1}{n}\sum_{k=1}^{n} 1 = \frac{1}{n}n = 1$$

Therefore, for every real number p with $1 \le p < \infty$, $\left(\frac{1}{n}\sum_{k=1}^{n} d(x_k, 2)\right)^p = 1^p = 1$. Summing both sides over n, we obtain

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(x_k, 2)\right)^p = \sum_{n=1}^{\infty} 1 = \infty$$

Thus, the sequence (x_n) is not an element of $ces_p(2)$ for every real number p with $1 \le p < \infty$.

Next, the focus is on $X = \mathbb{R}^2$, which means the objects of study are sequences of ordered pairs $\left(x_n^{(1)}, x_n^{(2)}\right)$ with $x_n^{(1)}, x_n^{(2)} \in \mathbb{R}$ for all $n \in \mathbb{N}$. With a given reference point in $\mathbb{R}^2$, an absolute Cesàro sequence space can be constructed in this way.

Example 3.4 Given the metric space $(\mathbb{R}^2, d)$ where d is the Euclidean metric. For each $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}$, the metric d is defined by $d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. Choose the reference point $(1,1) \in \mathbb{R}^2$. Then the absolute Cesàro-type sequence space built on $(\mathbb{R}^2, d)$ with reference point $(1,1)$ is is the collection of all sequences (x_n) with $x_n = \left(x_n^{(1)}, x_n^{(2)}\right)$ on $\mathbb{R}^2$ statisfying

$$ces_p((1,1)) = \left\{(x_n): \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(x_k, (1,1))\right)^p < \infty\right\}$$

Where p is a real number that statisfy $1 \le p < \infty$ and for $1 \le k \le n, n \in \mathbb{N}$, $d(x_k, (1,1)) \coloneqq \sqrt{\left(x_k^{(1)} - 1\right)^2 + \left(x_k^{(2)} - 1\right)^2}$.

Below are examples of sequences that fall within this space, as well as sequences that do not.

a. The sequence (x_n) defined by $x_n = \left(1 + \frac{(-1)^n}{5^n}, 1\right)$ is an element of $ces_p((1,1))$ for every real number p with $1 < p < \infty$.

Proof: Let $n \in \mathbb{N}$ be arbitrary. Notice that for each $k = 1,2,\dots,n$,

$$d(x_k,(1,1)) = \sqrt{\left(1 + \frac{(-1)^k}{5^k} - 1\right)^2 + (1-1)^2}$$
$$= \sqrt{\left(\frac{(-1)^k}{5^k}\right)^2} = \sqrt{\frac{(-1)^{2k}}{5^{2k}}} = \frac{1}{5^k}$$

Hence, for every $n \in \mathbb{N}$,

$$\frac{1}{n}\sum_{k=1}^{n} d(x_k,(1,1)) = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{5^k}$$

We know that $\sum_{k=1}^{\infty}\frac{1}{5^k}$ is a geometric series with ratio $\frac{1}{5}$, hence it converges to $\frac{1}{4}$. Therefore, for each $n \in \mathbb{N}$, $\sum_{k=1}^{n}\frac{1}{5^k} \le \sum_{k=1}^{\infty}\frac{1}{5^k} = \frac{1}{4}$, so $\frac{1}{n}\sum_{k=1}^{n}\frac{1}{5^k} \le \frac{1}{4n}, \forall n \in \mathbb{N}$. Now take any real number p with $1 < p < \infty$, then we have $\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{5^k}\right)^p \le \left(\frac{1}{4n}\right)^p = \frac{1}{4^p}\frac{1}{n^p}$, summing over n yields $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{5^k}\right)^p \le \frac{1}{4^p}\sum_{n=1}^{\infty}\frac{1}{n^p}$. By Theorem 2.2, $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges for $p > 1$. Hence, $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{5^k}\right)^p < \infty$.

Since $d(x_k,(1,1)) = \frac{1}{5^k}$, we conclude from the definition of $ces_p((1,1))$ that $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(x_k,(1,1))\right)^p < \infty.$ So the sequence (x_n) belongs to $ces_p((1,1))$ for every $p \in \mathbb{R}$ with $1 < p < \infty$.

b. The sequence (y_n) defined by $y_n = (n,n)$ is not an element of $ces_p((1,1))$ for every real number p with $1 \le p < \infty$.

Proof: Let n be any natural number. Note that for each $k = 1,2,\dots,n$,

$$d(y_k,(1,1)) = d((k,k),(1,1)) = \sqrt{(k-1)^2 + (k-1)^2}$$
$$= \sqrt{2(k-1)^2} = \sqrt{2}(k-1)$$

Hence, for every $n \in \mathbb{N}$,

$$\frac{1}{n}\sum_{k=1}^{n} d(y_k,(1,1)) = \frac{1}{n}\sum_{k=1}^{n}\sqrt{2}(k-1)$$

$$= \frac{\sqrt{2}}{n}\left(\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1\right)$$

$$= \frac{\sqrt{2}}{n}\left(\frac{n(n+1)}{2} - n\right)$$

$$= \sqrt{2}\left(\frac{n+1}{2} - 1\right)$$

$$= \frac{\sqrt{2}}{2}(n-1)$$

Therefore, for real number p with $1 \le p < \infty$, we have $\left(\frac{1}{n}\sum_{k=1}^{n} d(y_k,(1,1))\right)^p = \left(\frac{\sqrt{2}}{2}(n-1)\right)^p = \left(\frac{\sqrt{2}}{2}\right)^p (n-1)^p$. Summing over n yields $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(y_k,(1,1))\right)^p = \sum_{n=1}^{\infty}\left(\frac{\sqrt{2}}{2}\right)^p (n-1)^p$

Let $b_n = \left(\frac{\sqrt{2}}{2}\right)^p (n-1)^p$. Then (b_n) is an unbounded real sequence, hence it does not converge to 0. By the contrapositive of Theorem 2.2, the series $\sum_{n=1}^{\infty} b_n$ diverges. Consequently, $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(y_k,(1,1))\right)^p$

diverges, so $(y_n) = (n,n) \notin ces_p((1,1))$ for every real number p with $1 \le p < \infty$.

Let $C[0,1]$ denote the set of all real-valued continuous functions on $[0,1]$. The metric on $C[0,1]$ allows us to define the absolute-type Cesàro sequence space similarly to the preceding examples. The complete definition can be found in the following example.

Example 3.5 Let $(C[0,1], d)$ be a metric space. For each $f, g \in C[0,1]$, define the metric d by

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$$

If we choose the reference point $f_0(x) = 0 \in C[0,1]$, then the absolute-type Cesàro sequence space built over $(C[0,1], d)$ with reference point given by the constant function $f_0(x) = 0$ is $ces_p(0) = \left\{(f_n) : \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\sup_{x\in[0,1]}|f_k(x)|\right)^p < \infty\right\}$,

where p is a real number that statisfy $1 \le p < \infty$.

Next, we will give examples of sequences that belong to this sequence space.

a. The sequence of functions (f_n) defined by $f_n(x) = \frac{1}{n^2}\sin x$ for every $x \in [0,1]$ and every $n \in \mathbb{N}$, is an element of $ces_p(0)$ for every real number p with $1 < p < \infty$.

Proof: Let $n \in \mathbb{N}$ be arbitrary. Notice that for each $= 1,2,\dots,n$,

$$\sup_{x\in[0,1]}|f_k(x)| = \sup_{x\in[0,1]}\left|\frac{1}{k^2}\sin x\right| = \sup_{x\in[0,1]}\left|\frac{1}{k^2}\right||\sin x|$$
$$= \frac{1}{k^2}\sup_{x\in[0,1]}|\sin x|$$

Since $\sin x \ge 0$ and $\sin x$ is increasing on [0,1], we have $\sup_{x\in[0,1]}|\sin x| = \sup_{x\in[0,1]}\sin x = \sin 1$. Hence, $\sup_{x\in[0,1]}|f_k(x)| =$

$\frac{\sin 1}{k^2}$, and therefore for every $n \in \mathbb{N}$, $\frac{1}{n}\sum_{k=1}^{n} \sup_{x\in[0,1]} |f_k(x)| = \frac{1}{n}\sum_{k=1}^{n} \frac{\sin 1}{k^2} = \frac{\sin 1}{n}\sum_{k=1}^{n} \frac{1}{k^2}$.

Since the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, there exists a constant $0 < C < \infty$ such that $\sum_{k=1}^{\infty} \frac{1}{k^2} = C$. In particular, $\sum_{k=1}^{n} \frac{1}{k^2} = C$ for all $n \in \mathbb{N}$. Multiplying both sides by $\frac{\sin 1}{n} > 0$, we obtain $\frac{\sin 1}{n}\sum_{k=1}^{n} \frac{1}{k^2} \le \frac{C \sin 1}{n}$, let p be a real number statisfy $1 < p < \infty$, the we have $\left(\frac{\sin 1}{n}\sum_{k=1}^{n} \frac{1}{k^2}\right)^p \le \left(\frac{C \sin 1}{n}\right)^p = (C \sin 1)^p \frac{1}{n^p}$. Summing over n gives $\sum_{n=1}^{\infty}\left(\frac{\sin 1}{n}\sum_{k=1}^{n} \frac{1}{k^2}\right)^p \le \sum_{n=1}^{\infty}(C \sin 1)^p \frac{1}{n^p} = (C \sin 1)^p \sum_{n=1}^{\infty} \frac{1}{n^p}$.

Since $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for every $p > 1$, it follows that $(C \sin 1)^p \sum_{n=1}^{\infty} \frac{1}{n^p} < \infty$, and hence $\sum_{n=1}^{\infty}\left(\frac{\sin 1}{n}\sum_{k=1}^{n} \frac{1}{k^2}\right)^p < \infty$.

Finally, because $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} \sup_{x\in[0,1]} |f_k(x)|\right)^p = \sum_{n=1}^{\infty}\left(\frac{\sin 1}{n}\sum_{k=1}^{n} \frac{1}{k^2}\right)^p < \infty$, by the definition of $ces_p(0)$ we conclude that the function sequence (f_n) with $f_n(x) = \frac{1}{n^2}\sin x$ for $x \in [0,1]$ belongs to $ces_p(0)$ for every real number p with $1 < p < \infty$.

b. The sequence of function (f_n) defined by $f_n(t) = t^n$ for every $t \in [0,1]$ and every $n \in \mathbb{N}$ is not an element of $ces_p(0)$ for every real number p with $1 \le p < \infty$.

Proof: Let $n \in \mathbb{N}$ be arbitrary. Notice that for each $k = 1,2,\dots,n$ we have $d(f_k, f_0) = d(f_k, 0) = \sup_{t\in[0,1]} |f_k(t) - 0| = \sup_{t\in[0,1]} |f_k(t)| = \sup_{t\in[0,1]} |t^k|$. Since $0 \le t \le 1$, it follows that $0^k \le t^k \le 1^k$, hence $0 \le t^k \le 1$ for all $k = 1,2,\dots,n$. Moreover, at $t = 1$ we obtain $f_k(1) = 1^k = 1$. Therefore, $\sup_{t\in[0,1]} |t^k| = \sup_{t\in[0,1]} t^k = 1$, so for every $k = 1,2,\dots,n$ we have $\sup_{t\in[0,1]} |f_k(t)| = 1$.

Consequently, for all $n \in \mathbb{N}$ we have

$$\frac{1}{n}\sum_{k=1}^{n} \sup_{t\in[0,1]} |f_k(t)| = \frac{1}{n}\sum_{k=1}^{n} 1 = \frac{1}{n} n = 1$$

for any real number p with $1 \le p < \infty$ we obtain $\left(\frac{1}{n}\sum_{k=1}^{n} \sup_{t\in[0,1]} |f_k(t)|\right)^p = 1^p = 1$, hence

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} \sup_{t\in[0,1]} |f_k(t)|\right)^p = \sum_{n=1}^{\infty} 1 = \infty$$

It follows that the function sequence (f_n) with $f_n(t) = t^n$ on $[0,1]$ is not an element of $ces_p(0)$ for every real number p with $1 \le p < \infty$.

B. The Fundamental Properties of the Absolute-Type Cesaro Sequence Space on a Metric Space

Thus far absolute-type Cesàro sequence space $ces_p(x_0)$ has been constructed and illustrated with examples within several metric spaces. In case constructions like these, the membership condition for a sequence is based on the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(x_k, x_0)\right)^p$, thus the point $x_0 \in X$ becomes an integral part of the construction of $ces_p(x_0)$. Consequently, even before delving into the deeper properties and the structure of $ces_p(x_0)$, one must analyze how the selection of this reference point shapes $ces_p(x_0)$. This forms the basis for the consideration of Theorem 3.1.

Theorem 3.1 Let (X, d) be a metric space and let p be a real number with $1 \le p < \infty$. If $x_0, x_1 \in X$ with $x_0 \ne x_1$, then $ces_p(x_0) \ne ces_p(x_1)$.

Proof: Define the sequence $O = (o_n)_{n\in\mathbb{N}}$ in X by $o_n = x_0$ for every $n \in \mathbb{N}$.

Then $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(o_k, x_0)\right)^p = \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(x_0, x_0)\right)^p$.

Since $d(x_0, x_0) = 0$, it follows that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(o_k, x_0)\right)^p = \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} 0\right)^p = \sum_{n=1}^{\infty} 0 = 0 < \infty$$

Hence $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(o_k, x_0)\right)^p < \infty$, so $O = (x_0, x_0, \dots) \in ces_p(x_0)$. Next, consider whether O belongs to $ces_p(x_1)$. Since $x_1 \ne x_0$, we have $d(x_0, x_1) > 0$.

Thus, $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(o_k, x_1)\right)^p = \sum_{n=1}^{\infty}\left(d(x_0, x_1)\right)^p$.

Let (s_n) be the sequence of partial sums of the series $\sum_{n=1}^{\infty}\left(d(x_0, x_1)\right)^p$, where $s_n = \sum_{k=1}^{n}\left(d(x_0, x_1)\right)^p = n\left(d(x_0, x_1)\right)^p$. Then (s_n) is unbounded, hence diverges.

Consequently, the series $\sum_{n=1}^{\infty}\left(d(x_0, x_1)\right)^p = \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(o_k, x_1)\right)^p$ diverges, so the constant sequence $O = (x_0, x_0, \dots) \notin ces_p(x_1)$ for every real number p with $1 \le p < \infty$.

Since $O \in ces_p(x_0)$ but $O \notin ces_p(x_1)$, it follows that $ces_p(x_0) \ne ces_p(x_1)$.

After explaining this dependence, we proceed to define basic structure on $ces_p(x_0)$, as we will show in the following theorem.

Theorem 3.2 Let (X, d) be a metric space and let $p \in \mathbb{R}$ with $1 \le p < \infty$. Fix a point $x_0 \in X$ and define $ces_p(x_0)$ as the absolute-type Cesàro sequence space over (X, d) with reference point x_0. For any sequences $A = (a_n)$ and $B = (b_n)$ with $A, B \in ces_p(x_0)$, define the function $\rho_p: ces_p(x_0) \times$

$ces_p(x_0) \to \mathbb{R}$ by

$$\rho_p(A,B) := \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k)\right)^p\right)^{\frac{1}{p}}$$

Then $(ces_p(x_0), \rho_p)$ is a metric space.

Proof: To prove that $(ces_p(x_0), \rho_p)$ is a metric space, we first show that ρ_p is well-defined on $ces_p(x_0) \times ces_p(x_0)$.

Let $A = (a_n)$ and $B = (b_n)$ be arbitrary elements of $ces_p(x_0)$.

By definition,

$\rho_p(A,B) := \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k)\right)^p\right)^{\frac{1}{p}}$Thus $\rho_p(A,B)$ is well-defined if and only if the series $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k)\right)^p$converges.

Let n be arbitrary a natural number. For each $k = 1,2,\dots,n$, since d is a metric on X, the triangle inequality gives $d(a_k,b_k) \le d(a_k,x_0) + d(b_k,x_0)$. Summing over $k = 1,\dots,n$ and dividing by n, we obtain $\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k) \le \frac{1}{n}\sum_{k=1}^{n} d(a_k,x_0) + \frac{1}{n}\sum_{k=1}^{n} d(b_k,x_0)$.

Define $\alpha_n = \frac{1}{n}\sum_{k=1}^{n} d(a_k,x_0)$ and $\beta_n = \frac{1}{n}\sum_{k=1}^{n} d(b_k,x_0)$. Then, for every $n \in \mathbb{N}$, $\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k) \le \alpha_n + \beta_n$. Raising both sides to the power p with $1 \le p < \infty$ yields

$\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k)\right)^p \le (\alpha_n + \beta_n)^p$. Summing over n gives $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k)\right)^p \le \sum_{n=1}^{\infty}(\alpha_n + \beta_n)^p$ Taking the power $\frac{1}{p}$ on both sides, we obtain

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k)\right)^p\right)^{\frac{1}{p}} \le \left(\sum_{n=1}^{\infty}(\alpha_n + \beta_n)^p\right)^{\frac{1}{p}}$$

By Minkowski's inequality for sequences, $(\sum_{n=1}^{\infty}(\alpha_n + \beta_n)^p)^{\frac{1}{p}} \le (\sum_{n=1}^{\infty}\alpha_n^p)^{\frac{1}{p}} + (\sum_{n=1}^{\infty}\beta_n^p)^{\frac{1}{p}}$. Substituting the definitions of α_n and β_n, we get

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k)\right)^p\right)^{\frac{1}{p}} \le \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k,x_0)\right)^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(b_k,x_0)\right)^p\right)^{\frac{1}{p}}$$

Since $A \in ces_p(x_0)$, the first series on the right-hand side is finite, and since $B \in ces_p(x_0)$, the second series is also finite. Therefore, the right-hand side is finite, which implies $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k)\right)^p < \infty$. Hence $\rho_p(A,B)$ is well-defined for all $A, B \in ces_p(x_0)$.

Next, we show that ρ_p is a metric on $ces_p(x_0)$. By the definition of a metric space, it suffices to verify that ρ_p satisfies the following four properties.

a. For any $A = (a_n)$ and $B = (b_n)$ in $ces_p(x_0)$, $\rho_p(A,B) \ge 0$.

Proof. Take arbitrary $A = (a_n)$ and $B = (b_n)$ in $ces_p(x_0)$.Let n be any natural number.

Since $A, B \in ces_p(x_0)$, we have $a_k, b_k \in X$ for each $k = 1,2,\dots,n$. Because d is a metric on X, $d(a_k,b_k) \ge 0$ for all such k. Hence $\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k) \ge 0$. Raising both sides to the power p yields $\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k)\right)^p \ge 0$ and summing over n gives $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k)\right)^p \ge 0$ and taking the p-th root shows

$\rho_p(A,B) = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k)\right)^p\right)^{\frac{1}{p}} \ge 0.$

b. For any $A = (a_n)$ and $B = (b_n)$ in $ces_p(x_0)$, $\rho_p(A,B) = 0$ if and only if $A = B$.

Proof. ($\Rightarrow$) Let $A = (a_n)$ and $B = (b_n)$ be in $ces_p(x_0)$ and assume $\rho_p(A,B) = 0$. Then $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k)\right)^p = 0$. Each term in the series is nonnegative, hence for every $n \in \mathbb{N}$, $\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k)\right)^p = 0$, so $\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k) = 0$. Since $\frac{1}{n} \ne 0$, it follows that $\sum_{k=1}^{n} d(a_k,b_k) = 0$. As each $d(a_k,b_k) \ge 0$, we must have $d(a_k,b_k) = 0$ for every $k = 1,2,\dots,n$. Because d is a metric, $d(a_k,b_k) = 0$ implies $a_k = b_k$. Since n is arbitrary, $a_k = b_k$ for all $k \in \mathbb{N}$, hence $A = B$.

($\Leftarrow$) Conversely, assume $A = B$. Then $a_k = b_k$ for all $k \in \mathbb{N}$, so $d(a_k,b_k) = 0$ for all k. Therefore,

$$\rho_p(A,B) = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} 0\right)^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty} 0\right)^{\frac{1}{p}} = 0$$

Thus $\rho_p(A,B) = 0$ if and only if $A = B$.

c. For any $A = (a_n)$ and $B = (b_n)$ in $ces_p(x_0)$, $\rho_p(A,B) = \rho_p(B,A)$.

Proof. Since d is a metric, $d(a_k,b_k) = d(b_k,a_k)$ for all k. Substituting into the definition of ρ_p yields

$$\rho_p(A,B) = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k)\right)^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(b_k,a_k)\right)^p\right)^{\frac{1}{p}} = \rho_p(B,A)$$

d. For any $A = (a_n)$, $B = (b_n)$, and $C = (c_n)$ in $ces_p(x_0)$, we have $\rho_p(A,B) \le \rho_p(A,C) + \rho_p(C,B)$.

Proof. Let n be any arbitrary natural number. For each $k =$

$1,2,\dots,n$, by the triangle inequality for d, we have $d(a_k,b_k) \le d(a_k,c_k)+d(c_k,b_k)$. Summing over $k=1,\dots,n$ and dividing by n gives

$$\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k) \le \frac{1}{n}\sum_{k=1}^{n} d(a_k,c_k) + \frac{1}{n}\sum_{k=1}^{n} d(c_k,b_k)$$

Define $u_n := \frac{1}{n}\sum_{k=1}^{n} d(a_k,c_k)$, dan $v_n := \frac{1}{n}\sum_{k=1}^{n} d(c_k,b_k)$ Then $\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k) \le u_n + v_n$. Raising both sides to the power p and summing over n yields

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k)\right)^p \le \sum_{n=1}^{\infty}(u_n+v_n)^p$$

Taking the p-th root gives $\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k)\right)^p\right)^{\frac{1}{p}} \le \left(\sum_{n=1}^{\infty}(u_n+v_n)^p\right)^{\frac{1}{p}}$

By Minkowski's inequality we have $\left(\sum_{n=1}^{\infty}(u_n+v_n)^p\right)^{\frac{1}{p}} \le \left(\sum_{n=1}^{\infty} u_n^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} v_n^p\right)^{\frac{1}{p}}$. Substituting back the definitions of u_n and v_n yields

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k,b_k)\right)^p\right)^{\frac{1}{p}} \le \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k,c_k)\right)^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(c_k,b_k)\right)^p\right)^{\frac{1}{p}}$$

That is, $\rho_p(A,B) \le \rho_p(A,C) + \rho_p(C,B)$.

From (i)–(iv), ρ_p is a metric on $ces_p(x_0)$. Hence $\left(ces_p(x_0),\rho_p\right)$ is a metric space.

In Theorem 3.2 it was shown that $\left(ces_p(x_0),\rho_p\right)$ is a metric space. Therefore, the notation $ces_p(x_0)$ is the absolute-type Cesàro sequence space constructed over the metric space (X, d) with a point of reference $x_0 \in X$, and ρ_p the metric on $ces_p(x_0)$ as described in Theorem 3.2. Now we focus on the inclusion relations for the $ces_p(x_0)$ for various p, as we state the following theorem.

Theorem 3.3 Let (X,d) be a metric space and let $x_0 \in X$. If $p,q \in \mathbb{R}$ satisfy $1 \le p < q < \infty$, then $ces_p(x_0) \subseteq ces_q(x_0)$.

Proof: Let (x_n) be any member of $ces_p(x_0)$. Since (x_n) is an element of $ces_p(x_0)$, it is a sequence in X, hence $x_n \in X$ for every $n \in \mathbb{N}$. By the definition of $ces_p(x_0)$, we have $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(x_k,x_0)\right)^p < \infty$. Let n be arbitrary natural number. Because d is a metric on X and $x_k, x_0 \in X$ for all $k = 1,2,\dots,n$, we have $d(x_k,x_0) \ge 0$. Consequently, $\frac{1}{n}\sum_{k=1}^{n} d(x_k,x_0) \ge 0$. For any real number p with $1 \le p < \infty$, we have $\left(\frac{1}{n}\sum_{k=1}^{n} d(x_k,x_0)\right)^p \ge 0$. Thus, each term in the series above is nonnegative. Since the series converges, every term is bounded above by the total sum. In other words, for each $n \in \mathbb{N}$ $\left(\frac{1}{n}\sum_{k=1}^{n} d(x_k,x_0)\right)^p \le \sum_{m=1}^{\infty}\left(\frac{1}{m}\sum_{k=1}^{m} d(x_k,x_0)\right)^p$Taking the p-th root on both sides gives $\frac{1}{n}\sum_{k=1}^{n} d(x_k,x_0) \le \left(\sum_{m=1}^{\infty}\left(\frac{1}{m}\sum_{k=1}^{m} d(x_k,x_0)\right)^p\right)^{\frac{1}{p}}$. Since $(x_n) \in ces_p(x_0)$, the series $\sum_{m=1}^{\infty}\left(\frac{1}{m}\sum_{k=1}^{m} d(x_k,x_0)\right)^p$converges. Hence, by the definition of convergence of a series, there exists a real number S with $0 \le S < \infty$ such that $S = \sum_{m=1}^{\infty}\left(\frac{1}{m}\sum_{k=1}^{m} d(x_k,x_0)\right)^p$. Therefore, for every $n \in \mathbb{N}$, $\frac{1}{n}\sum_{k=1}^{n} d(x_k,x_0) \le S^{\frac{1}{p}}$. Because $S \ge 0$ and $1 \le p < \infty$, the real number $S^{\frac{1}{p}}$ is well-defined. Set $M := S^{\frac{1}{p}}$. Then $\frac{1}{n}\sum_{k=1}^{n} d(x_k,x_0) \le M, \forall n \in \mathbb{N}$.

Now note that $p < q$ implies $q - p > 0$. Hence, we get $\left(\frac{1}{n}\sum_{k=1}^{n} d(x_k,x_0)\right)^{q-p} \le M^{q-p}, \forall n \in \mathbb{N}$. Multiplying both sides by $\left(\frac{1}{n}\sum_{k=1}^{n} d(x_k,x_0)\right)^p \ge 0$ gives $\left(\frac{1}{n}\sum_{k=1}^{n} d(x_k,x_0)\right)^q \le M^{q-p}\left(\frac{1}{n}\sum_{k=1}^{n} d(x_k,x_0)\right)^p$. Summing over n we obtain

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(x_k,x_0)\right)^q \le M^{q-p}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(x_k,x_0)\right)^p$$

But the right-hand side is finite because $M < \infty$ and $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(x_k,x_0)\right)^p = S < \infty$.Hence,

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(x_k,x_0)\right)^q < \infty$$

which shows $(x_n) \in ces\_q(x_0)$. Since (x_n) was taken arbitrary in $ces_p(x_0)$, it follows that $ces_p(x_0) \subseteq ces_q(x_0)$.

This implies that for all sequences $A \in ces_p(x_0)$ also belongs to $ces_q(x_0)$. Consequently, for all $A,B \in ces_p(x_0)$, $\rho_q(A,B)$ is well-defined. Next, we analyze the distance A and B obtain using ρ_p and ρ_q. In fact, the distance induced by ρ_q is never greater than that induced by ρ_p. This is stated in the following theorem.

Theorem 3.4 Let (X,d) be a metric space and let $x_0 \in X$. If $p,q \in \mathbb{R}$ satisfy $1 \le p < q < \infty$ and $A = (a_n), B = (b_n) \in ces_p(x_0)$, then $\rho_q(A,B) \le \rho_p(A,B)$.

Proof: Let $A = (a_n)$ and $B = (b_n)$ arbitrary member of $ces_p(x_0)$. Since $A,B \in ces_p(x_0)$, both are sequences in X, so

$a_n, b_n \in X$ for every $n \in \mathbb{N}$. By the definition of $ces_p(x_0)$, we have $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k, x_0)\right)^p < \infty$ and $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(b_k, x_0)\right)^p < \infty$. Let n be any natural number. Since d is a metric on X and $a_k, b_k, x_0 \in X$, for each $k = 1,2,\dots,n$ we have $d(a_k, b_k) \geq 0$, hence $\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k)\right)^p \geq 0$. Moreover, by the triangle inequality, $d(a_k, b_k) \leq d(a_k, x_0) + d(x_0, b_k), \forall k = 1,2,\dots,n$ Summing over k and dividing by n gives $\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k) \leq \frac{1}{n}\sum_{k=1}^{n} d(a_k, x_0) + \frac{1}{n}\sum_{k=1}^{n} d(x_0, b_k)$. Define, for each $n \in \mathbb{N}$, $u_n := \frac{1}{n}\sum_{k=1}^{n} d(a_k, x_0)$ and $v_n := \frac{1}{n}\sum_{k=1}^{n} d(x_0, b_k)$. Then $u_n, v_n \geq 0$ and $\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k) \leq u_n + v_n$. Raising both sides to the power p yields $\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k)\right)^p \leq (u_n + v_n)^p$. Summing over n we obtain $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k)\right)^p \leq \sum_{n=1}^{\infty}(u_n + v_n)^p$. Taking the power $\frac{1}{p}$ on both sides gives

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k)\right)^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty}(u_n + v_n)^p\right)^{\frac{1}{p}}$$

By Minkowski's inequality applied to the sequences (u_n) and (v_n),

$$\left(\sum_{n=1}^{\infty}(u_n + v_n)^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} u_n^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} v_n^p\right)^{\frac{1}{p}}$$

Substituting the definitions of u_n and v_n yields

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k)\right)^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k, x_0)\right)^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(x_0, b_k)\right)^p\right)^{\frac{1}{p}}$$

Since $A, B \in ces_p(x_0)$, the two series on the right are finite; hence the left-hand side is finite as well. In particular, $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k)\right)^p < \infty$. Next, observe that the series defining $\rho_p(A, B)$ has nonnegative terms. Therefore, for every $n \in \mathbb{N}$, $\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k)\right)^p \leq \sum_{m=1}^{\infty}\left(\frac{1}{m}\sum_{k=1}^{m} d(a_k, b_k)\right)^p = \left(\rho_p(A, B)\right)^p$.Taking the power $\frac{1}{p}$ gives $\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k) \leq \rho_p(A, B), \forall n \in \mathbb{N}$. Now take arbitrary $p, q \in \mathbb{R}$ with $1 \leq p < q < \infty$. Since $q - p > 0$ and $\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k) \geq 0$, $\rho_p(A, B) \geq 0$, raising the last inequality to the power $q - p$ yields $\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k)\right)^{q-p} \leq \rho_p(A, B)^{q-p}$. Multiplying both sides by $\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k)\right)^p$ gives $\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k)\right)^{q-p+p} \leq \rho_p(A, B)^{q-p}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k)\right)^p$. Summing over n we obtain

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k)\right)^q \leq \rho_p(A, B)^{q-p}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k)\right)^p$$

But by definition, $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k)\right)^p = \rho_p(A, B)^p$ Hence, $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k)\right)^q \leq \rho_p(A, B)^q$. Taking the power $\frac{1}{q}$ on both sides gives

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k)\right)^q\right)^{\frac{1}{q}} \leq \rho_p(A, B)$$

By the definition of ρ_q, the left-hand side is exactly $\rho_q(A, B)$. Therefore, $\rho_q(A, B) \leq \rho_p(A, B)$.

From Theorem 3.2, it follows that $\left(ces_p(x_0), \rho_p\right)$ is a metric space. From neighborhood, convergence, Cauchy sequences, and completeness definitions in previous definitions 2.2, 2.3, 2.4, and 2.5, we proceed to define these concepts in $\left(ces_p(x_0), \rho_p\right)$.

Definition 3.2 Let $Y \in ces_p(x_0)$. For a real number $\varepsilon > 0$, the neighborhood of Y with radius ε, denoted by $N(Y; \varepsilon)$, is the set

$$N(Y; \varepsilon) = \left\{X \in ces_p(x_0) : \rho_p(X, Y) < \varepsilon\right\}$$

Definition 3.3 A sequence $(Y_m)_{m\in\mathbb{N}}$ in $ces_p(x_0)$, where $Y_m = \left(y_m^{(k)}\right)_{k\in\mathbb{N}}$, is said to converge to some $U \in ces_p(x_0)$ if for every real number $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that for every $m \geq K$, $Y_m \in N(U; \varepsilon)$, or equivalently, $\rho_p(Y_m, U) < \varepsilon$.

Definition 3.4 A sequence $(Y_m)_{m\in\mathbb{N}}$ in $ces_p(x_0)$, where $Y_m = \left(y_m^{(k)}\right)_{k\in\mathbb{N}}$, is called a Cauchy sequence if for every real number $\varepsilon > 0$ there exists $H \in \mathbb{N}$ such that for all $m, q \geq H$, $\rho_p(Y_m, Y_q) < \varepsilon$.

Definition 3.5 The metric space $\left(ces_p(x_0), \rho_p\right)$ is called complete if every Cauchy sequence in $ces_p(x_0)$ converges to some $U \in ces_p(x_0)$ with respect to the metric ρ_p.

Following the definition of completeness for the metric space $\left(ces_p(x_0), \rho_p\right)$ in Definition 3.4, we will analyze the interdependencies between the completeness of the original space (X, d) and the completeness of $\left(ces_p(x_0), \rho_p\right)$. This is captured in the following Theorem 3.5.

Theorem 3.5 If (X, d) is a complete metric space, then $(ces_p(x_0), \rho_p)$ is complete for every real number p with $1 \leq p < \infty$.

Proof: Let $(Y_m)_{m\in\mathbb{N}}$ be an arbitrary Cauchy sequence in $(ces_p(x_0), \rho_p)$, where $Y_m = \left(y_m^{(k)}\right)_{k\in\mathbb{N}}$. Fix $\varepsilon > 0$ and $n \in \mathbb{N}$. Since $(Y_m)_{m\in\mathbb{N}}$ is Cauchy in $(ces_p(x_0), \rho_p)$, for $\frac{\varepsilon}{n} > 0$ there exists $N\left(\frac{\varepsilon}{n}\right) \in \mathbb{N}$ such that for all $m, q \geq N\left(\frac{\varepsilon}{n}\right)$,

$$\rho_p(Y_m, Y_q) = \left(\sum_{r=1}^{\infty}\left(\frac{1}{r}\sum_{k=1}^{r} d\left(y_m^{(k)}, y_q^{(k)}\right)\right)^p\right)^{\frac{1}{p}} < \frac{\varepsilon}{n}$$

Raising both sides to the power p gives $\sum_{r=1}^{\infty}\left(\frac{1}{r}\sum_{k=1}^{r} d\left(y_m^{(k)}, y_q^{(k)}\right)\right)^p < \left(\frac{\varepsilon}{n}\right)^p$. Because each term in the series is nonnegative, for every $t \in \mathbb{N}$ we have

$$\left(\frac{1}{t}\sum_{k=1}^{t} d\left(y_m^{(k)}, y_q^{(k)}\right)\right)^p \leq \sum_{r=1}^{\infty}\left(\frac{1}{r}\sum_{k=1}^{r} d\left(y_m^{(k)}, y_q^{(k)}\right)\right)^p < \left(\frac{\varepsilon}{n}\right)^p$$

Taking $t = n$ yields $\left(\frac{1}{n}\sum_{k=1}^{n} d\left(y_m^{(k)}, y_q^{(k)}\right)\right)^p < \left(\frac{\varepsilon}{n}\right)^p$ and since both sides are nonnegative, $\frac{1}{n}\sum_{k=1}^{n} d\left(y_m^{(k)}, y_q^{(k)}\right) < \frac{\varepsilon}{n}$. Multiplying by n gives $\sum_{k=1}^{n} d\left(y_m^{(k)}, y_q^{(k)}\right) < \varepsilon$. Because each summand is nonnegative, in particular $d\left(y_m^{(n)}, y_q^{(n)}\right) \leq \sum_{k=1}^{n} d\left(y_m^{(k)}, y_q^{(k)}\right) < \varepsilon$.Thus, for each fixed $n \in \mathbb{N}$, the sequence $\left(y_m^{(n)}\right)_{m\in\mathbb{N}} = \left(y_1^{(n)}, y_2^{(n)}, y_3^{(n)}, \dots\right)$ is a Cauchy sequence in (X, d). Since (X, d) is complete, for every $n \in \mathbb{N}$ there exists $u_n \in X$ such that $y_q^{(n)} \to u_n$ as $q \to \infty$. Define $U = (u_1, u_2, u_3, \dots)$. Next, fix $\varepsilon > 0$. Because $(Y_m)_{m\in\mathbb{N}}$ is Cauchy in $(ces_p(x_0), \rho_p)$, there exists $N(\varepsilon) \in \mathbb{N}$ such that for all $m, q \geq N(\varepsilon)$, $\rho_p(Y_m, Y_q) < \varepsilon$. Raising to the power p gives

$$\sum_{r=1}^{\infty}\left(\frac{1}{r}\sum_{k=1}^{r} d\left(y_m^{(k)}, y_q^{(k)}\right)\right)^p < \varepsilon^p$$

For any fixed $m \geq N(\varepsilon)$ and any $r \in \mathbb{N}$, since $y_q^{(k)} \to u_k$ as $q \to \infty$, we have $d\left(y_m^{(k)}, y_q^{(k)}\right) \to d\left(y_m^{(k)}, u_k\right)$. Hence $\frac{1}{r}\sum_{k=1}^{r} d\left(y_m^{(k)}, y_q^{(k)}\right) \to \frac{1}{r}\sum_{k=1}^{r} d\left(y_m^{(k)}, u_k\right)$ as $q \to \infty$. Taking the limit $q \to \infty$ in the inequality above yields

$$\sum_{r=1}^{\infty}\left(\frac{1}{r}\sum_{k=1}^{r} d\left(y_m^{(k)}, u_k\right)\right)^p < \varepsilon^p$$

Taking the p-th root, we obtain

$$\left(\sum_{r=1}^{\infty}\left(\frac{1}{r}\sum_{k=1}^{r} d\left(y_m^{(k)}, u_k\right)\right)^p\right)^{\frac{1}{p}} < \varepsilon$$

that is, $\rho_p(Y_m, U) < \varepsilon$ for all $m \geq N(\varepsilon)$. So $Y_m \to U$ in ρ_p. It remains to show that $U \in ces_p(x_0)$, i.e. $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(u_k, x_0)\right)^p < \infty$. Fix $n \in \mathbb{N}$. By the triangle inequality in (X, d), for each $k = 1,2,\dots,n$, $d(u_k, x_0) \leq d\left(u_k, y_m^{(k)}\right) + d\left(y_m^{(k)}, x_0\right)$. Summing over $k = 1,\dots,n$ and dividing by n gives

$$\frac{1}{n}\sum_{k=1}^{n} d(u_k, x_0) \leq \frac{1}{n}\sum_{k=1}^{n} d\left(u_k, y_m^{(k)}\right) + \frac{1}{n}\sum_{k=1}^{n} d\left(y_m^{(k)}, x_0\right)$$

Set $\alpha_n := \frac{1}{n}\sum_{k=1}^{n} d\left(u_k, y_m^{(k)}\right)$ and $\beta_n := \frac{1}{n}\sum_{k=1}^{n} d\left(y_m^{(k)}, x_0\right)$. Then $\frac{1}{n}\sum_{k=1}^{n} d(u_k, x_0) \leq \alpha_n + \beta_n$, so

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(u_k, x_0)\right)^p \leq \sum_{n=1}^{\infty}(\alpha_n + \beta_n)^p$$

Taking p-th roots and using Minkowski's inequality yields

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(u_k, x_0)\right)^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty}\alpha_n^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}\beta_n^p\right)^{\frac{1}{p}}$$

Now define the constant sequence $O = (o_n)$ in X by $o_n = x_0$ for all n. Then $O \in ces_p(x_0)$, since $d(o_k, x_0) = d(x_0, x_0) = 0$ implies $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(o_k, x_0)\right)^p = 0 < \infty$. Moreover $\left(\sum_{n=1}^{\infty}\alpha_n^p\right)^{\frac{1}{p}} = \rho_p(U, Y_m)$ dan $\left(\sum_{n=1}^{\infty}\beta_n^p\right)^{\frac{1}{p}} = \rho_p(Y_m, O)$, Hence

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(u_k, x_0)\right)^p\right)^{\frac{1}{p}} \leq \rho_p(U, Y_m) + \rho_p(Y_m, O)$$

Since $Y_m \to U$ in ρ_p, choose m large enough so that $\rho_p(U, Y_m) < 1$. Also, because $Y_m \in ces_p(x_0)$ and $O \in ces_p(x_0)$, we have $\rho_p(Y_m, O) < \infty$. Therefore, $\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(u_k, x_0)\right)^p\right)^{\frac{1}{p}} \leq 1 + \rho_p(Y_m, O) < \infty$, which implies $\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(u_k, x_0)\right)^p\right)^{\frac{1}{p}} < \infty$. Thus $U \in ces_p(x_0)$, and every Cauchy sequence in $(ces_p(x_0), \rho_p)$ converges to an element of $ces_p(x_0)$. Consequently, $(ces_p(x_0), \rho_p)$ is complete.

Considering such a case when (X, d) is not complete and understanding how it explains the non-completeness of $(ces_p(x_0), \rho_p)$ is part of the next theorem.

Theorem 3.6 If (X, d) is not complete, then $(ces_p(x_0), \rho_p)$ is not complete for every real number p with $1 < p < \infty$.

Proof: Since (X, d) is not complete, there exists a Cauchy sequence $(x_m)_{m\in\mathbb{N}}$ in X, but $(x_m)_{m\in\mathbb{N}}$ does not converge in X.

To show that $(ces_p(x_0), \rho_p)$ is not complete for every real number p with $1 < p < \infty$, we construct a Cauchy sequence $(Y_m)_{m\in\mathbb{N}}$ in $ces_p(x_0)$ that does not converge in $ces_p(x_0)$. For each $m \in \mathbb{N}$, define $Y_m = (y_1^{(m)}, y_2^{(m)}, \dots) = (y_k^{(m)})_{k\in\mathbb{N}}$ by $y_k^{(m)} = \begin{cases} x_m, & k = 1 \\ x_0, & k \geq 2 \end{cases}$. Thus $Y_m = (x_m, x_0, x_0, \dots)$. Fix $n \in \mathbb{N}$. If $n = 1$, then $\frac{1}{n}\sum_{k=1}^{n} d(y_k^{(m)}, x_0) = d(x_m, x_0)$. If $n \geq 2$, then $y_1^{(m)} = x_m$ and $y_k^{(m)} = x_0$ for $k \geq 2$, hence $\frac{1}{n}\sum_{k=1}^{n} d(y_k^{(m)}, x_0) = \frac{1}{n}(d(x_m, x_0) + 0) = \frac{1}{n} d(x_m, x_0)$
Therefore,

$$\frac{1}{n}\sum_{k=1}^{n} d(y_k^{(m)}, x_0) = \begin{cases} d(x_m, x_0), n = 1 \\ \dfrac{1}{n} d(x_m, x_0), n \geq 2 \end{cases}$$

Let $1 < p < \infty$. Then $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(y_k^{(m)}, x_0)\right)^p = (d(x_m, x_0))^p + \sum_{n=2}^{\infty}\left(\frac{1}{n} d(x_m, x_0)\right)^p = (d(x_m, x_0))^p\left[1 + \sum_{n=2}^{\infty}\frac{1}{n^p}\right]$. Since $p > 1$, the series $\sum_{n=2}^{\infty}\frac{1}{n^p}$ converges. Moreover, because $(x_m)_{m\in\mathbb{N}}$ is Cauchy in (X, d), it is bounded, so there exists $M > 0$ such that $d(x_m, x_0) \leq M$ for all $m \in \mathbb{N}$. Hence

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(y_k^{(m)}, x_0)\right)^p \leq M^p\left(1 + \sum_{n=2}^{\infty}\frac{1}{n^p}\right) < \infty$$

Thus $Y_m \in ces_p(x_0)$ for every $m \in \mathbb{N}$.

Next, we prove that $(Y_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $(ces_p(x_0), \rho_p)$. Take arbitrary $m, n \in \mathbb{N}$ and $r \in \mathbb{N}$. Since $y_1^{(m)} = x_m$, $y_1^{(n)} = x_n$ dan $y_k^{(m)} = y_k^{(n)} = x_0$ for all $k \geq 2$, we have $d(y_k^{(m)}, y_k^{(n)}) = \begin{cases} d(x_m, x_n), k = 1 \\ 0, \ k \geq 2 \end{cases}$. Therefore, $\frac{1}{r}\sum_{k=1}^{r} d(y_k^{(m)}, y_k^{(n)}) = \frac{1}{r} d(x_m, x_n)$. By the definition of ρ_p,

$$(\rho_p(Y_m, Y_n))^p = \sum_{r=1}^{\infty}\left(\frac{1}{r} d(x_m, x_n)\right)^p = \sum_{r=1}^{\infty}\frac{1}{r^p}(d(x_m, x_n))^p = (d(x_m, x_n))^p\sum_{r=1}^{\infty}\frac{1}{r^p}$$

Since $p > 1$, the series $\sum_{r=1}^{\infty}\frac{1}{r^p}$ converges; let $S = \sum_{r=1}^{\infty}\frac{1}{r^p} > 0$, Then $(\rho_p(Y_m, Y_n))^p = (d(x_m, x_n))^p S$, hence $\rho_p(Y_m, Y_n) = ((d(x_m, x_n))^p S)^{\frac{1}{p}} = d(x_m, x_n) S^{\frac{1}{p}}$. Set $K = S^{\frac{1}{p}} > 0$. Thus $\rho_p(Y_m, Y_n) = K d(x_m, x_n)$. Given $\varepsilon > 0$, define $\varepsilon^* := \frac{\varepsilon}{K} > 0$. Because $(x_m)_{m\in\mathbb{N}}$ is Cauchy in (X, d), there exists $N \in \mathbb{N}$ such that for all $m, n \geq N, d(x_m, x_n) < \varepsilon^*$. Consequently, for all $m, n \geq N$, $\rho_p(Y_m, Y_n) = K d(x_m, x_n) < K\varepsilon^* = \varepsilon$. Hence $(Y_m)_{m\in\mathbb{N}}$ is Cauchy in $(ces_p(x_0), \rho_p)$.

Now assume, for contradiction, that $(ces_p(x_0), \rho_p)$ is complete. Then $(Y_m)_{m\in\mathbb{N}}$ converges in $ces_p(x_0)$; that is, there exists $Z = (z_k)_{k\in\mathbb{N}} \in ces_p(x_0)$ such that $Y_m \to Z$ in ρ_p. Thus, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m \geq N$, $\rho_p(Y_m, Z) < \varepsilon$. By definition,

$$(\rho_p(Y_m, Z))^p = \sum_{r=1}^{\infty}\left(\frac{1}{r}\sum_{k=1}^{r} d(y_k^{(m)}, z_k)\right)^p$$

Since each term in this series is nonnegative, for any $t \in \mathbb{N}$ we have
$\left(\frac{1}{t}\sum_{k=1}^{t} d(y_k^{(m)}, z_k)\right)^p \leq \sum_{r=1}^{\infty}\left(\frac{1}{r}\sum_{k=1}^{r} d(y_k^{(m)}, z_k)\right)^p = (\rho_p(Y_m, Z))^p$.Taking p-th roots gives $\frac{1}{t}\sum_{k=1}^{t} d(y_k^{(m)}, z_k) \leq \rho_p(Y_m, Z), \forall t \in \mathbb{N}$. In particular, for $t = 1$, $d(y_1^{(m)}, z_1) \leq \rho_p(Y_m, Z)$. Because $y_1^{(m)} = x_m$, we obtain $d(x_m, z_1) \leq \rho_p(Y_m, Z)$. Since $\rho_p(Y_m, Z) \to 0$, it follows that $d(x_m, z_1) \to 0$, i.e., $x_m \to z_1$ in (X, d). This contradicts the choice of $(x_m)_{m\in\mathbb{N}}$ as a nonconvergent Cauchy sequence in X. Therefore $(ces_p(x_0), \rho_p)$ is not complete for every $1 < p < \infty$.

The previous theorem proves that if (X, d) is not complete, then for any real number p such that $1 < p < \infty$, $(ces_p(x_0), \rho_p)$ is also not complete. However, the case $p = 1$ is different, because the membership condition for $ces_1(x_0)$ only holds for the constant sequence whose all terms are the reference point x_0. This is pointed out in the following proposition.

Proposition 3.1 Let (X, d) be any metric space and let $x_0 \in X$. Then $ces_1(x_0) = \{(x_n): x_n = x_0, \forall n \in \mathbb{N}\}$.

Proof: Let (X, d) be a metric space and let $x_0 \in X$. First observe that the constant sequence $X_0 = (x_0, x_0, x_0, \dots)$ belongs to $ces_1(x_0)$. Indeed, for every $n \in \mathbb{N}$, $\frac{1}{n}\sum_{k=1}^{n} d(x_0, x_0) = 0$, so $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(x_0, x_0)\right) = 0 < \infty$. Next, let (x_n) be arbitrary member of $ces_1(x_0)$. By the definition of $ces_1(x_0)$,

$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d(x_k, x_0)\right) < \infty$. For each $n \in \mathbb{N}$, define $S_n := \sum_{k=1}^{n} d(x_k, x_0)$. Since $d(x_k, x_0) \geq 0$ for all k, we have $S_n \geq 0$ and (S_n) is nondecreasing. The membership condition above can be written as $\sum_{n=1}^{\infty}\frac{S_n}{n} < \infty$.We claim that $S_n = 0$ for every $n \in \mathbb{N}$. Suppose not. Then there exists $n_0 \in \mathbb{N}$ such that $S_{n_0} > 0$. Because (S_n) is nondecreasing, $S_n \geq S_{n_0}$ for all $n \geq n_0$. Hence

$$\sum_{n=n_0}^{\infty}\frac{S_n}{n} \geq \sum_{n=n_0}^{\infty}\frac{S_{n_0}}{n} = S_{n_0}\sum_{n=n_0}^{\infty}\frac{1}{n}$$

which diverges, a contradiction to $\sum_{n=1}^{\infty}\frac{S_n}{n} < \infty$. Therefore $S_n = 0$ for all n. Since $S_n = \sum_{k=1}^{n} d(x_k, x_0) = 0$ and each term $d(x_k, x_0) \geq 0$, it follows that $d(x_k, x_0) = 0$ for every $k \leq n$. Because n is arbitrary, we obtain $d(x_k, x_0) = 0$ for all $k \in \mathbb{N}$. As d is a metric, $d(x_k, x_0) = 0$ implies $x_k = x_0$ for all k. Thus

(x_n) is the constant sequence $(x_0, x_0, \dots)$. Consequently, $ces_1(x_0) = \{(x_n): x_n = x_0, \forall n \in \mathbb{N}\}$.

Based on the previous proposition, we know that $ces_1(x_0)$ has only one member, which is the constant sequence $(x_0, x_0, x_0, \dots)$. Since a singleton space is always complete for any metric, it follows that $(ces_1(x_0), \rho_1)$ is complete metric space even when (X, d) is not complete.

4. Conclusion

This study builds an absolute type Cesàro sequence space over a metric space (X, d) for an arbitrary reference point $x_0 \in X$ and defines $ces_p(x_0) \coloneqq \left\{(x_n): \sum_{n=1}^{\infty} \left(\frac{1}{n}\sum_{k=1}^{n} d(x_k, x_0)\right)^p < \infty\right\}$, $1 \leq p < \infty$. A metric ρ_p is defined on this space by $\rho_p(A, B) \coloneqq \left(\sum_{n=1}^{\infty} \left(\frac{1}{n}\sum_{k=1}^{n} d(a_k, b_k)\right)^p\right)^{\frac{1}{p}}$, $\forall A = (a_n), B = (b_n) \in ces_p(x_0)$.The results establish that for all $1 \leq p < \infty$, $\left(ces_p(x_0), \rho_p\right)$ becomes a metric space. If $1 \leq p < q < \infty$, then $ces_p(x_0) \subseteq ces_q(x_0)$ and $\rho_q(A, B) \leq \rho_p(A, B)$ for all $A, B \in ces_p(x_0)$. Finally, the metric space $\left(ces_p(x_0), \rho_p\right)$ is complete if only if (X, d) is complete, and it is not complete if only if (X, d) is not complete.

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