Some Results on Semi groups when *f*-Prime Ideals are Maximal

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***Abstract*: In this research paper it is verified that S becomes f-primary if proper f-prime ideals are maximal in S. It is verified that for S being quasi-commutative, right cancellative and S is either a f-primary (or) S being a semi group with f-semi primary ideals are f-primary then proper f-prime ideals are maximal. We proved that S being cancellative and commutative with either S is f-primary (or) an ideal Q is f - primary in S ⇔ $r_f(Q)$ is a f-prime ideal, and therefore the proper f-prime ideals in S are maximal. It is verified that for S is quasi-commutative and right cancellative having identity, then these statements are equivalent. (1) proper f-prime ideals are maximal. (2) S being f-primary (3) f-semi primary ideals are f-primary. (4) If g & h are not the units in S, then ∃ n, m∈N ∋ g^n = hs and h^m = gr for any s, r ∈ S. It is shown that if S is a quasi-commutative and right cancellative without identity then these conditions are equivalent (1) S being f- primary (2) f-semi primary ideals are f- primary (3) There are no proper f-prime ideals in S (4) g and h are not units in S, then ∃ n, m∈N ∋ g^n = hs and h^m = gr for any s, r ∈ S. It is verified that for S being quasi-commutative and right cancellative then these statements (1) S is f-primary (2) f-semi primary ideals are f-primary (3) proper f-prime ideals are maximal, are equivalent.**

***Keywords*: f - prime & f - semiprime ideals, f-primary & f-semi primary ideals, maximal ideal.**

1. Introduction

"*The algebraic theory of semigroups"* was introduced by *Clifford* and *Preston* [5], [6]; *Petrich* [7] "*Structure and ideal theory of semigroups"* & "*Primary ideals in semigroups"* were presented by *Anjaneyulu.A* [1][2] "*A generalization of prime ideals in semigroups"* was presented by *Hyekyung Kim* [3] "*generalization of prime ideals in rings"* was introduced by *Murata. K*, *Kurata. Y* and *Murabayashi. H* [8] "*prime and maximal ideals in semigroups"* was presented by *Scwartz. S* [4] "*Commutative primary semigroups"* was developed by M. Satyanarayana [9]. "*f-primary ideals in semigroups*", "*f-semiprime ideal in semigroups*" and "*f-prime radical in semigroups*" were developed by *T. Radha Rani, A. Gangadhara Rao* [10]-[12].

2. Preliminaries

Definition 2.1: Let (*S*,.) be a set and *S*≠ ∅ If '.' Is a binary operation on *S* and it holds associative then *S* is defined as a "*Semigroup*".

Note 2.2: Throughout this paper *S* will indicate a semigroup**.**

Definition 2.3: If *qr=rq* ∀ *q,r*∈*S* then *S* is called as "*commutative"*

Definition 2.4: If *qs* = *s*∀*s* ∈*S* then the element *q* in *S* is called as "*left identity"* of *S*.

Definition 2.5: If *sq* = *s* ∀*s*∈*S* then the element *q* in *S* is called as "*right identity"* of *S*.

Definition 2.6: An element *q* in *S* is both left and right identity in *S* so it is called as "*identity".*

Definition 2.7: Let *Q*(≠ ∅) is a set in *S*. *Q* is entitled as "*left ideal*" in *S* when *SQ*⊆*Q***.**

Definition 2.8: Let *Q*(≠ ∅) is a set in *S*. *Q* is entitled as "*right ideal*" in *S* when *QS*⊆*Q*.

Definition 2.9: A subset *Q* in *S* is both left and right ideal in *S* then it is known as "*ideal*" in *S*.

Definition 2.10: The intersection of each one of the ideals in *S* carrying a non-void set *P* is known as the "*ideal generated by P"*. It is signified as <*P*>.

Definition 2.11: Some ideal *Q* of *S* is called as "*principal ideal"* if *Q* is an ideal generated by single element set. On the off chance that an ideal *Q* is generated by *q*, at that point *Q* is indicated as <*q*> or *J*[*q*]

Definition 2.12: Some ideal *Q* of *S* is called as "*completely prime ideal"* given *k,l* ∈*Q*, *kl* ∈*Q*, either *k* ∈*Q* or *l* ∈*Q***.**

Definition 2.13: Some ideal *D* in *S* is known as "*prime ideal"* when *Q, R* be ideals of *S*,

QR⊆*D* infers either *Q*⊆*D* or *R*⊆*D*.

Definition 2.14: Let *P* be some ideal in *S*, then the intersection of each one of the prime ideals carrying *P* is said to be "*prime radical"* or just "*radical of P*" and it is meant by $\sqrt{P}$ or *rad P*.

Definition 2.15: Let *P* be some ideal in *S*, then the intersection of each one of the completely prime ideals carrying

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P is entitled as "*complete prime radical*" or "*complete radical*" of $P$and indicated as "*c.rad P*".

Definition 2.16: Some ideal K in S is known to be "*completely semiprime*" if $k \in S$, $k \in K$ for some $n \in N \Rightarrow k^n \in K$.

Theorem 2.17: An ideal K of S is completely semiprime$\Leftrightarrow$ $k \in S$, $k^2 \in K$ implies $k \in K$.

Definition 2.18: Some ideal K in S is known to be "*semiprime*" if X is an ideal of S, $X^n \subseteq K$ for some $n \in N \Rightarrow X \subseteq K$.

Theorem 2.19: An ideal K of S is semiprime $\Leftrightarrow X$ is an ideal of S, $X^2 \subseteq K \Rightarrow X \subseteq K$.

Theorem 2.20: If $f(w)$ is an ideal in S then $f(W) = \bigcup_{w\in W} f(w)$ is an ideal**.**

Definition 2.21: A Subset K of S is named as *"p-system"* $\Leftrightarrow \langle k\rangle\langle l\rangle \cap Q \neq \emptyset$for any
k, l in K.

Definition 2.22: A Subset K of S is known as *"sp-system"* $\Leftrightarrow \langle k\rangle^2 \cap Q \neq \emptyset$for any $k \in K$.

Definition 2.23: For any $f \in F$ a subset K of S is known as an *"f-system"* $\Leftrightarrow$ it consists of a *p-system* $K^* \ni K^* \cap f(k) \neq \emptyset$for each $k \in K$.

Definition 2.24: For any $f \in F$ a subset K of S is known as an *"sf-system"* $\Leftrightarrow$ it consists of a *sp-system* $K^* \ni K^* \cap f(k) \neq \emptyset$for each $k \in K$.

Definition 2.25: A proper ideal J in S is called *"f-prime"* $\Leftrightarrow$its complement J^c is a *f-system.*

Definition 2.26: A proper ideal J in S is called *"f-semiprime"*$\Leftrightarrow$its complement J^c a *sf-system.*

Theorem 2.27: Let J be some ideal in S then J^c is a *f-system* $\Rightarrow f(j_1)\, f(j_2) \subseteq J \Rightarrow j_1 \in J$or $j_2 \in J$.

Theorem 2.28: Let J be some ideal in S then J^c is a *f-system* $\Rightarrow f(K)\, f(R) \subseteq J \Rightarrow f(K) \subseteq J$ or $f(R) \subseteq J$.

Corollary 2.29: Let J be a f-prime ideal in S*.* Then for any two ideals K, R in S such that $f(K)\, f(R) \subseteq J$ implies either $f(K) \subseteq J$ or $f(R) \subseteq J$**.**

Theorem 2.30: Let J bean f-prime ideal of S, so the subsequent statements are equivalent.

(i) $f(k)\, f(r) \subseteq J \Rightarrow k \in J$ or $r \in J$.

(ii) $f(K)\, f(R) \subseteq J \Rightarrow f(K) \subseteq J$ or $f(R) \subseteq J$, for all ideals K, R of S.

Theorem 2.31: Let J be some ideal in S then J^c is a *sf*-system$\Rightarrow f(j_1)^2 \subseteq J \Rightarrow j_1 \in J$.

Theorem 2.32: Let J be some ideal in S then J^c is a *sf*-system$\Rightarrow f(K)^2 \subseteq J \Rightarrow f(K) \subseteq J$.

Here $f(K) = \bigcup_{k\in K} f(k)$

Corollary 2.33: Let J be some ideal of S. J is called as *"f-semiprime ideal"* if K is some ideal in S. $f(K)^2 \subseteq J$ implies $f(K) \subseteq J$. here $f(K) = \bigcup_{k\in K} f(k)$.

Theorem 2.34: An ideal J of S is f-semiprime$\Leftrightarrow$for any $k \in S$ if $f(k)\, S\, f(k) \subseteq J \Rightarrow k \in J$.

Definition 2.35: Let J be some ideal in S then "*f-rad J*" = $\{x/K \cap J \neq \emptyset$for each f-system K containing $x\}$is the *f-radical* of J and is specified by $r_f(J)$.

Definition 2.36: Let J be some ideal in S then "*sf-rad J*" = $\{x/K \cap J \neq \emptyset$for each *sf*-system K containing $x\}$is the *sf-radical* of J and is specified by $r_{sf}(J)$.

Theorem 2.37: *Let J be some ideal in S then sf- rad J* $=\bigcap_{J\subseteq P_i} P_i$, Where P_i is *f-semiprime* ideal in S containing J.

Theorem 2.38: *Let G be some ideal in S then* $r_f(G) = \bigcap_{G\subseteq P_i} P_i, 1 \leq i \leq n$ *Where* P_i *is f-prime ideal in S.*

Theorem 2.39: Let G and H be two ideals of S. If $G \subseteq H$, then $r_f(G) \subseteq r_f(H)$.

Definition 2.40: Q is both left and right f-primary ideal implies Q is "*f - primary ideal.*"

Definition 2.41: An ideal A of S is said to be *sf-primary* if $r_f(Q)$is f-prime ideal.

Definition 2.42: S is said to be *sf-Primary semigroup* if every ideal of S is a *sf -Primary* ideal.

Theorem 2.43: Let J be some ideal in S then *sf- rad J* $=\bigcap_{J\subseteq P_i} P_i$, Where P_i is *f-semiprime* ideal in S containing J.

Theorem 2.44: Let G be some ideal in S then $r_f(G) = \bigcap_{G\subseteq P_i} P_i, 1 \leq i \leq n$Where P_i *is f*-prime ideal in S.

Theorem 2.45: An ideal J in S is f-semiprime ideal$\Leftrightarrow r_f(J) = J$.

Corollary 2.46: An ideal J of S is a f-semiprime ideal$\Leftrightarrow J$ is the intersection of all f-prime ideals of S contains J.

Corollary 2.47: If J is an ideal in S, then $r_f(J)$ is the smallest semiprime ideal of S.

Theorem2.48: Let S has identity and J be a unique maximal ideal in S.If $r_f(W) = J$ for any ideal W in S, then W is a f-primary ideal.

3. Main Results

Theorem 3.1: Let *S* be a semigroup and it contains identity. If (non-zero, assume this if *S* has zero) proper *f*-prime ideals are maximal in *S*, then *S* is a *f*-primary semigroup.

Proof: If S is not a simple semigroup with identity, then S contains a unique maximal ideal J, it is the union of proper ideals in S.

Suppose G be a (nonzero) proper ideal in S.

Then $r_f(G) = J$.

From the theorem 2.48, G is a f-primary ideal.

Suppose S contains zero and $< 0 >$ be a f-prime ideal.

Then $< 0 >$ is f-primary and consequently S remains f-primary.

Suppose $< 0 >$ should not be a f-prime ideal.

Then $r_f(< 0 >) = J$

Therefore, from the theorem 2.48, $< 0 >$ be a f-primary ideal.

Hence S is f-primary.

Example 3.2: Consider $S = \{u, v, w, 1\}$ is the semigroup by the operation multiplication and it is given in below table.

.	*u*	*v*	*w*	1
u	*u*	*u*	*u*	*u*
v	*u*	*v*	*v*	*v*
w	*u*	*v*	*w*	*w*
1	*u*	*v*	*w*	1

Here S is f-primary, the f-prime ideal $<u>$ not being a maximal ideal.

Theorem 3.3: Let S be a right cancellative quasi commutative semigroup. If S is a f-primary semigroup or a semigroup in which f-semiprimary ideals are f-primary, then for any f-primary ideal W, $r_f(W)$ is non maximal implies $W = r_f(W)$ and it is f-prime.

***Proof*:** Let S be right cancellative and quasi commutative.

Since $r_f(W)$ is not maximal, then $\exists$ an ideal Q in $S \ni r_f(W) \subset Q \subset S$.

Suppose $q \in Q \setminus r_f(W)$ and $t \in r_f(W)$.

Then $W \subseteq W \cup <qt> \subseteq r_f(W)$.

By the theorem 2.39, $r_f(W) \subseteq r_f(W \cup <qt>) \subseteq r_f(r_f(W))$.

Therefore $r_f(W \cup <qt>) = r_f(W)$.

$\therefore$ By assumption $W \cup <qt>$ is a f-primary ideal.

Suppose $s \in S \setminus Q$.

Then $qst \in W \cup <qt>$.

Since $q \in r_f(W) = r_f(W \cup <qt>)$ and $W \cup <qt>$ is a f-primary ideal, then $st \in W \cup <qt>$.

If $st \in <qt>$, then $st = rqt$ for certain $r \in S$.

Therefore, by right cancellative property, we get $s = rq \in Q$, which is a contradiction.

$\therefore$ $st \in W \Rightarrow s \notin r_f(W)$ and $t \in r_f(W)$. Hence $W = r_f(W)$ and W is f-prime.

Theorem 3.4: Let S be a right cancellative quasi commutative semigroup. If S is either f-primary or a semigroup in which f-semi primary ideals are f-primary then proper f-prime ideals are maximal in S.

***Proof*:** Suppose that J is not a maximal ideal.

Consider $\mathscr{L} = S \setminus J$ and $Q = \{q \in S \,/\, qu \in <h> \text{for any } u \in \mathscr{L}\}$.

Obviously, Q is an ideal in S.

If $q \in Q$, then $qu \in <h> \subseteq J \Rightarrow q \in J$.

Therefore $Q \subseteq J$.

Suppose $v \in J$ and $G = \{v^k u \,/\, u \in \mathscr{L} \text{ and } k \text{ is whole number}\}$.

Since $vu \in G$ and $vu \notin \mathscr{L}$, then G is a sub-semigroup containing $\mathscr{L}$ properly.

By the assumption, J be a minimal f-prime ideal containing $<h>$ and $\mathscr{L}$ is a maximal sub-semigroup not meeting $<h>$.

Since $\mathscr{L} \subset G$ and $G \cap <h> \neq \emptyset$, then $\exists\, k \in N \ni v^k u \in <h> \Rightarrow v^k \in Q$.

Therefore $v \in r_f(Q)$.

Now $J \subseteq r_f(Q) \subseteq r_f(J) = J \Rightarrow J = r_f(Q)$.

Then by our supposition Q being a f-primary ideal.

Since J is not a maximal ideal, then by Theorem 3.4, $J = Q$.

Now J is also a minimal f-prime ideal containing $<h^2>$.

Suppose $\mathscr{B} = \{w \in S \,/\, wu \in <h^2> \text{ for any } u \in \mathscr{L}\}$.

In the similar way, we get $\mathscr{B} = J$.

Since $h \in J = Q = \mathscr{B}$, then $hu = sh^2$ for any $s \in S^1$.

Since S is quasi commutative, then $hu = u^n h$ for some $n \in N$.

By the right cancellative property, $u^n = sh \in <h>$. It is a conflict.

Therefore, J be a maximal ideal.

If J is some proper f-prime ideal, then for any $h \in J$ a minimal f-prime ideal contains $<h>$.

Hence J is a maximal ideal.

Corollary 3.5: Let S be a cancellative commutative semigroup. Then S is a f-primary semigroup or an ideal Q in S is f-primary if and only if $r_f(Q)$ is a f-prime ideal and the proper f-prime ideals are maximal in S.

***Proof*:** Since S being commutative, then $u, v \in S$, $uv = vu$

Now $uv = v^1 u \Rightarrow uv = v^n u$, for $n = 1$, that represents S is quasi commutative.

Therefore, S is cancellative and quasi commutative.

Hence by the theorem 3.4, the proof of the corollary is clear.

Theorem 3.6: If S is a quasi-commutative right cancellative semigroup and it contains identity, then the following constrains are equivalent.

1) Proper f-prime ideals are maximal in S.

2) S is a f-primary semigroup

3) f-semi primary ideals are f-primary in S.

4) If g and h are not units in S, then there exists some natural numbers n and m such that $g^n = hs$ and $h^m = gr$ for some $s, r \in S$.

***Proof*:** By the theorems 3.1 and 3.4, we have (1), (2) and (3) are equivalent.

Suppose (1) that is proper f-prime ideals are maximal in S.

Since S contains identity, then a unique maximal ideal exists in S, say J.

Clearly J is the only f-prime ideal.

Suppose g & h are not the units in S and $<g>$ and $<h>$ are ideals in S.

Then $r_f(<g>)$, $r_f(<h>)$ are proper f-prime ideals. Since proper f-prime ideals are maximal in S, then we get $g, h \in J$.

Therefore $r_f(<g>) = r_f(<h>) = J$.

$\therefore \exists\, n, m \in N \ni g^n = hs$ and $h^m = gr$ for any $s, r \in S$.

Thus (1) $\Rightarrow$ (4).

Assume (4): Suppose Q is a f-semiprimary ideal in S.

Let $<q>, <l>$ are any two ideals in $S \ni <q><l> \subseteq Q$ and $l \notin Q$.

Then by the hypothesis $q \in r_f(Q)$.

Therefore, Q is left f-primary.

In the same way, we can show that Q will be right f-primary.

Therefore, Q being f-primary.

Hence (4) $\Rightarrow$ (3).

Theorem 3.7: If S is a quasi-commutative right cancellative semigroup and it does not contain identity, then the following constrains are equivalent.

1) S is f-primary.

2) f-semi primary ideals become f-primary in S.

3) There are no proper f-prime ideals in S.

4) If g and h are not units in S, then there exists $n, m \in N$ such that $g^n = hs$ and $h^m = gr$ for any $s, r \in S$.

***Proof*:** Let S be quasi commutative and right cancellative without the identity.

Then (1) $\Rightarrow$ (2) is clear.

(1) $\Rightarrow$ (2): Assume that f-semiprimary ideals are f-primary in S.

By the Theorem 3.5, proper f-prime ideals become maximal in S.

Let Q be some f-prime ideal.

If Q is maximal, then $S \setminus Q$ be a group.

Let u be an identity element in $S \setminus Q$.

Then u becomes an idempotent in S.

Since S is right cancellative, then u is the right identity in S.

Since S is quasi commutative, then idempotents is commute in S.

Therefore, u is the identity in S, which is a conflict.

$\therefore$ S does not contain f-prime ideals.

Thus (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (4): Assume that there are no proper f-prime ideals in S.

Then for some ideal W in S, $r_f(W) = S$.

Suppose $k, l \in S$.

Now $r_f(<k>) = r_f(<l>) = S$, by this (4) follows.

By the theorem 3.6, (4) $\Rightarrow$ (1) is clear.

Corollary 3.8: Let *S* be quasi commutative and right cancellative semigroup. Then the following constrains are equivalent.

1) *S* is *f*-primary

2) *f*-semi primary ideals are *f*-primary in *S*.

3) Proper *f*-prime ideals become maximal in *S*.

Moreover, there are no idempotents in *S* excluding the identity.

***Proof*:** By the theorem 3.4, S is f-primary or a semigroup such that where the f-semi primary ideals become f-primary.

Then proper f-prime ideals become maximal in S.

Therefore, by the theorems 3.6 and 3.7, the proof of this theorem is clear.

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