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Convergence and Statistically Convergence in the Usual Metric Space ℝ

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***Abstract*: The convergence of sequences of real numbers in metric spaces is a fundamental concept widely utilized in various problem-solving and mathematical development contexts. A sequence is said to converge to a real number x if its elements approach x as they tend to infinity. In 1951, the concept of convergence was extended to include statistical convergence. A sequence is termed statistically convergent to a real number x if the proportion of its elements approaching x tends to one as the elements tend to infinity. Any sequence that converges in the usual metric space R is also statistically convergent with the same limit. Despite ongoing advancements in convergence theory, necessary conditions for the ordinary convergence of sequences in the usual metric space R have yet to be established. Consequently, this article discusses the relationship between ordinary and statistical convergence in the usual metric space R. This research explores the interplay among three convergence concepts, aiming to introduce a novel approach for determining whether a sequence converges. One of the theorems found is that if a sequence is convergent, it is also statistically convergent; however, the converse does not hold. A statistically convergent sequence will be convergent if it is monotone.**

***Keywords*: Convergent sequence, Statistical convergence.**

1. Introduction

A sequence of real numbers is said to converge to a real number x if the sequence approaches x as its elements tend to infinity [1]. This concept of convergence has continued to evolve. One of the developments in the study of sequence convergence is the concept of statistical convergence of sequences of real numbers. The concept of statistical convergence was first discussed by Zygmund in 1935 and was later formally introduced by Fast [2]. Unlike convergent sequences, a statistically convergent sequence to a real number x does not require all its elements to approach x; rather, the proportion of elements that approach x must tend to one as the elements tend to infinity [3]. Numerous studies have been conducted to further explore the development of statistical convergence. The concept of statistical convergence in G-metric spaces has also been introduced by previous researchers ([6]-[10]). However, these studies have not detailed the relationship between ordinary convergence and statistical convergence, particularly within the usual metric space ℝ.

2. Literature Survey

A. Metric Spaces

Definition 2.1. [8] *A metric space is a pair (X, d) where X is a non-empty set and d is a function defined on X x X such that for every $a, b, c \in X$ satisfies the following axiom*

1. $d(a, b) \geq 0$
2. $d(a, b) = 0$ *if and only if* $a = b$
3. $d(a, b) = d(b, a)$
4. $d(a, c) \leq d(a, b) + d(b, c)$

Throughout this article, the metric space used is the usual metric space $(\mathbb{R}, d)$, where for every $x, y \in \mathbb{R}$ the distance d is defined as $d(x, y) = |x - y|$.

B. Convergences

Definition 2.2. [9] *A real number a is called the limit of a sequence $A = (a_n)$ if for every real number $\varepsilon > 0$, there exists a natural number $j = j(\epsilon)$ such that for every natural number $n \geq j$ the inequality $|a_n - a| < \varepsilon$ holds. This is denoted as*

$$\lim A = \lim_{n \to \infty}(a_n) = \lim(a_n) = a$$

A sequence is said to be convergent if it has a limit. If a sequence does not have a limit, then it is not a convergent sequence.

Theorem 2.1. [9] (Uniqueness of the Limit) A convergent sequence of real numbers has at most one limit.

Definition 2.3. [11] *A sequence of real numbers $X = (x_n)$ is called a Cauchy sequence if for every real number $\epsilon > 0$ there exists a natural number $h = h(\epsilon)$ such that for all natural numbers n, m with $n, m \geq h$ then*

$$|x_n - x_m| < \epsilon$$

Theorem 2.2. [9] Given a sequence of real numbers $X = (x_n)$. If the sequence (x_n) converges, then (x_n) is a Cauchy sequence.

C. Statistical Convergences

To discuss the definition of statistical convergence, the concept of the cardinality of a set must first be understood. The number of elements in a set A s called the cardinality of set A and is denoted by $|A|$or $n(A)$.

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Definition 2.4. [12] *Given $K \subset \mathbb{N}$. the Natural Density or Asymptotic Density of K is defined as follows:*

$$\delta(K) = \lim_{n\to\infty}\left(\frac{1}{n}|\{k \in K: k \le n\}|\right)$$

where $|\{k \in K: k \le n\}|$ denotes the cardinality of the set $\{k \in K: k \le n\}$.

Definition 2.5. [13] *A sequence of real numbers $X = (x_n)$ is said to be statistically convergent to a real number L if for every real number $\varepsilon > 0$, the following holds:*

$$\lim_{n\to\infty}\left(\frac{1}{n}|\{k \le n: |x_k - L| \ge \varepsilon\}|\right) = 0$$

and is denoted as

$$st - \lim(x_n) = L.$$

Theorem 2.3. [14] *Given a sequence of real numbers (x_n). If $st - lim(x_n) = x$ and $st - lim(x_n) = y$ then $x = y$.*

Definition 2.6. [15] *A sequence $X = (x_n)$ is said to be a statistically Cauchy sequence if for every real number $\varepsilon > 0$, there exists a natural number N such that*

$$\lim_{n\to\infty}\left(\frac{1}{n}|\{k \le n: |x_k - x_N| \ge \varepsilon\}|\right) = 0$$

3. Main Result

Several properties related to statistical convergence are encapsulated in the following theorem.

Theorem 3.1. *If a sequence converges to x, hen it converges statistically to x.*

Proof. Given a sequence (x_n) that converges to x. This means that for every real number $\varepsilon > 0$ there exists $j \in \mathbb{N}$ such that for every natural number $k \ge j$ then $|x_k - x| < \varepsilon$. Let us form a set A_j as follows:

$$A(j) = \{k \in \mathbf{N}: k \ge j, |x_k - x| < \varepsilon\}$$

It is clear that

$$|A(j)| = \infty$$

Since there exists $j \in \mathbb{N}$ such that for every $k \ge j$ berlaku $|x_k - x| < \varepsilon$ then the number of $k \in \mathbb{N}$ for which $|x_{n_0} - x| \ge \varepsilon$ is at most $j - 1$. Therefore, we have:

$$\lim_{n\to\infty}\left(\frac{1}{n}|\{k \le n: |x_k - x| \ge \varepsilon\}|\right) < \lim_{n\to\infty}\left(\frac{j}{n}\right) = 0.$$

Thus, it is proven that (x_n) converges statistically to x.

The converse of the above theorem does not hold because not all statistically convergent sequences are convergent. Below, we provide an example of a statistically convergent sequence that is not convergent.

Example 3.1. Given a sequence of real numbers (x_n) with

$$x_n = \begin{cases} 1, & n = k^2, k = 1,2,3,\cdots \\ 0, & \text{lainnya} \end{cases}$$

To prove that this sequence converges statistically, the proof will be divided into two cases.
Given any real number $\varepsilon > 0$

Case 1, for $\varepsilon > 1$.
When $\varepsilon > 1$, then $|\{k \in \mathbb{N}: |x_k - 0| \ge \varepsilon\}| = 0$, because for every $k \in N$, $|x_k - 0| < \varepsilon$. Thus, it is clear that

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{1}{n}|\{k \le n: |x_k - 0| \ge \varepsilon\}|\right) &= \lim_{n\to\infty}\left(\frac{1}{n}|\varnothing|\right) \\ &= \lim_{n\to\infty}\left(\frac{0}{n}\right) \\ &= 0\end{aligned}$$

Case 2, for $0 < \varepsilon \le 1$.
When $\varepsilon \le 1$, then $\{k \in \mathbb{N}: |x_k - 0| \ge \varepsilon\} = \{1,4,9,16,25,\dots\}$. Thus,

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{1}{n}|\{k \le n: |x_k - 0| \ge \varepsilon\}|\right) &= \lim_{n\to\infty}\left(\frac{1}{n}|\{1,4,9,16,,\dots\}|\right) \\ &= \lim_{n\to\infty}\left(\frac{\lfloor\sqrt{n})\rfloor}{n}\right) \\ &\le \lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\right) \\ &= 0\end{aligned}$$

Thus, it is demonstrated that the sequence (x_n) converges statistically to 0.
To prove that the sequence does not converge, consider the following. From the given formulas, we have two subsequences of (x_n), $X = (x_n = 1)$ and $Y = (y_n = 0)$ with $\lim X = 1 \neq 0 = \lim Y$.

Several conditions for a statistically convergent sequence to also be convergent are outlined in the following theorems.

Theorem 3.2. *Given a sequence (x_n) statistically converging to x. If for every real number $\varepsilon > 0$, $|A_\varepsilon| = |\{k \in \mathbb{N}: |x_k - x| \ge \varepsilon\}| < \infty$, then the sequence converges to x.*

Proof. Consider any real number $\varepsilon > 0$. Since the sequence (x_n) statistically converges to x, it follows that for $\varepsilon > 0$, then

$$\lim_{n\to\infty}\left(\frac{1}{n}|k \le n: |x_k - x| \ge \varepsilon|\right) = 0.$$

Given that

$$|A_\varepsilon| = |\{k \in \mathbb{N}: |x_k - x| \ge \varepsilon\}| < \infty.$$

Then for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that for all natural numbers $i \ge N(\varepsilon)$ then $|x_i - x| < \epsilon$. Hence, it is proven that the sequence (x_n) converges to x.

Example 3.2. The sequence $(x_n) = (\frac{(-1)^n}{n})$ converges to 0.
Proof: Consider any real number $\varepsilon > 0$. Let k be the largest natural number such that $k \le \frac{1}{\varepsilon}$, then

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{1}{n}|\{k \le n: |x_k - 0| \ge \varepsilon\}|\right) &= \lim_{n\to\infty}\left(\frac{1}{n}|\{1,2,3,..\mathrm{k}\}|\right) \\ &= \lim_{n\to\infty}\left(\frac{k}{n}\right) \\ &= \mathrm{k}\lim_{n\to\infty}\left(\frac{1}{n}\right) \\ &= 0\end{aligned}$$

Therefore, (x_n) converges statistically to 0.
Because

$$|A_\epsilon| = |\{k \le n: |x_k - 0| \ge \varepsilon\}| = |\{1,2,3,..\mathrm{k}\}| = \mathrm{k} < \infty$$

according to Theorem 3.2, the sequence (x_n) converges to 0.

Theorem 3.3. *A sequence (x_n) that converges statistically to x will converge to x if and only if the sequence is monotonic. Furthermore:*

(i) If $X = (x_n)$ is a monotonically increasing sequence and converges statistically to x, then

$$lim(x_n) = x \geq x_k \text{ for every } k \in \mathbb{N}$$

(ii) If $X = (x_n)$ is a monotonically decreasing sequence and converges statistically to x, then

$$lim(x_n) = x \leq x_k \text{ for every } k \in \mathbb{N}$$

Proof. Since every convergent sequence always converges statistically, it remains to prove that the sequence (x_n) converges.

Let $\varepsilon > 0$ be any real number. Since the sequence (x_n) converges statistically to x, it holds for the real number $\varepsilon > 0$ that

$$\lim_{n\to\infty}\left(\frac{1}{n}|k \leq n: |x_k - x| \geq \varepsilon|\right) = 0.$$

(i) For the case of a monotonically increasing sequence (x_n):

Since (x_n) is monotonically increasing, we have $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$.

Jika $|x_k - x| \geq \varepsilon$ for some $k \in \mathbb{N}$, then for every $i < k$, $|x_i - x| \geq \varepsilon$, because

$$\lim_{n\to\infty}\left(\frac{1}{n}|\{k \leq n: |x_k - x| \geq \varepsilon\}|\right) = 0,$$

This means for the real number $\varepsilon > 0$, there exists $N = k + 1 \in \mathbb{N}$ such that for every $j \in \mathbb{N}, j \geq \mathbb{N}$ then $|x_j - x| < \varepsilon$ and $\lim(x_n) = x \geq x_k$ for every $k \in \mathbb{N}$.

(ii) For the case of a monotonically decreasing sequence (x_n) :

A monotonically decreasing sequence (x_n) means $x_1 \geq x_2 \geq \cdots \geq x_n \geq x_{n+1} \geq \cdots$.

Jika $|x_k - x| \geq \varepsilon$ for some $k \in \mathbb{N}$, then for every $i > k$, $|x_i - x| \geq \varepsilon$ because

$$\lim_{n\to\infty}\left(\frac{1}{n}|\{k \leq n: |x_k - x| \geq \epsilon\}|\right) = 0.$$

This implies that for the real number $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for every $j \in \mathbb{N}$ with $j \geq M$ then $|x_j - x| < \varepsilon$ and $\lim(x_n) = x \leq x_k$ for every $k \in \mathbb{N}$.

Example 3.3. The sequence $(x_n) = \left(\frac{4}{5}, \frac{5}{5}, \frac{5{,}5}{5}, \frac{5{,}75}{5}, \ldots\right)$ is an convergent sequence to $\frac{3}{2}$.

Proof: Clearly, the sequence (x_n) is monotonically increasing because $x_{n+1} > x_n$. Next, it will be proven that (x_n) converges statistically to $\frac{3}{2}$. Let $\varepsilon > 0$ be arbitrary. The proof is divided into two cases:

If $\varepsilon > \frac{1}{2}$, then $|\{k \in \mathbb{N}: |x_k - \frac{3}{2}| \geq \varepsilon\}| = 0$, because for every $k \in \mathbb{N}$, $|x_k - \frac{3}{2}| < \varepsilon$. Thus,

$$\lim_{n\to\infty}\left(\frac{1}{n}\left|\left\{k \leq n: \left|x_k - \frac{3}{2}\right| \geq \varepsilon\right\}\right|\right) = \lim_{n\to\infty}\left(\frac{1}{n}|\varnothing|\right) = \lim_{n\to\infty}\left(\frac{0}{n}\right) = 0.$$

If $0 < \varepsilon \leq \frac{1}{2}$, then

$$\lim_{n\to\infty}\frac{1}{n}(|\{k \leq n: |x_k - 0| \geq \varepsilon\}|) \leq \lim_{n\to\infty}\left(\frac{\left(\frac{1}{2\varepsilon}\right)}{n}\right) = \frac{1}{2\varepsilon}\lim_{n\to\infty}\left(\frac{1}{n}\right) = \frac{1}{2\varepsilon}.0 = 0$$

Thus, it is proven that (x_n) converges statistically to $\frac{3}{2}$.

Since (x_n) is monotonic and statistically converges to $\frac{3}{2}$, by Theorem 3.3, (x_n) converges to $\frac{3}{2}$.

Theorem 3.4. *If (x_n) converges statistically, then (x_n) is a statistically Cauchy sequence.*

Proof. Let $\varepsilon > 0$ be arbitrary. Suppose $\text{st} - lim(x_n) = x$. Since ϵ is a real number and $\varepsilon > 0$, then $\frac{\varepsilon}{2}$ is also a real number and $\frac{\varepsilon}{2} > 0$. Hence, the set $A_\varepsilon = \{n \in \mathbb{N}: |x_n - x| \geq \frac{\varepsilon}{2}\}$ has an asymptotic density of 0 and $B_\varepsilon = \{n \in \mathbb{N}: |x_n - x| < \frac{\varepsilon}{2}\}$ has an asymptotic density of 1. If we choose $m \in \mathbb{N}$ such that $|x_m - x| < \frac{\varepsilon}{2}$, then

$$\begin{aligned}|x_n - x_m| &= |x_n - x + x - x_m| \\ &\leq |x_n - x| + |x_m - x| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Therefore, the set $\{n \in \mathbb{N}: |x_n - x_m| < \varepsilon\}$ has an asymptotic density of 1, or equivalently, $\{n \in \mathbb{N}: |x_n - x_m| \geq \varepsilon\}$ has an asymptotic density of 0, which means (x_n) is a statistically Cauchy sequence.

Corollary 3.2. *If the sequence (x_n) is a Cauchy sequence, then (x_n) is statistically Cauchy.*

Theorem 3.5. *If (x_n) statistically converges to L, then there exists $A \subset \mathbb{N}$ with $\delta(A) = 1$, such that the subsequence (x_m) of (x_n) where $m \in A$ converges to L.*

Proof. Let $\varepsilon > 0$ be any real number. Since (x_n) statistically converges to L, for this ε, we have

$$\lim_{n\to\infty}\left(\frac{1}{n}|\{k \leq n: |y_k - L| \geq \varepsilon\}|\right) = 0$$

Choosing $A = \{n \in \mathbb{N}: |x_n - L| < \varepsilon\}$, it is evident that $\delta(A) = 1$. Therefore, if we select the subsequence (x_m) of (x_n) with $m \in A$ it follows that $|x_m - L| < \varepsilon$, or in other words, (x_m) converges to L.

4. Conclusion

By examining the concepts and properties of ordinary convergence and statistical convergence in the usual metric, several conditions were identified under which a statistically convergent sequence in the usual metric is also an convergent sequence. Since this article focuses solely on convergence and statistical convergence in the usual metric space $\mathbb{R}$, there is an opportunity to conduct research in broader metric spaces, such as G-metric spaces, and on more general sequences, such as sequences of complex numbers or function sequences. Additionally, other types of convergence could be explored to determine if there are relationships that could provide new approaches for identifying whether a sequence converges.

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