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Properties of Compact Set in G-metric Space

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***Abstract*: In mathematical analysis, topology is built by metric spaces. A metric space is a set in which the metric axioms are satisfied. Many mathematicians discuss the generalization of metric spaces. One of them is the concept of G −metric space, denoted by (X, G), which was introduced in 2006. Within the metric space, there are many special sets that have played an important role in developments in the field of mathematical analysis, in particular compact sets. A set is said to be compact if each open cover of the set has finite subcovers. The properties of compact sets have been discussed in metric spaces, Hausdorff spaces, topological spaces, and fuzzy metric spaces. However, there are no researchers who discuss the properties of compact sets in G −metric spaces. Therefore, to expand the discussion of the concept of G −metric space and the properties of compact sets that apply to it, this article discusses the proof of theorems related to the properties of compact sets in G −metric space. Compact sets in G −metric spaces have the properties of being closed and bounded. However, not all closed and bounded sets are compact sets. To prove the compactness of a set in G −metric space, in addition to using the concept of open covers, it can also be proven by the G −completeness and G −totally boundedness properties of a set.**

***Keywords*: G −Metric Space, Compact Set, Topology.**

1. Introduction

Mathematical analysis is one of the branches of pure mathematics, one of which is related to the concept of neighborhood and set, where there is a set called compact set. A mathematician from France named Henri Lebesgue introduced compact sets for the first time in 1902 [1]. The concept of compact sets has a very important role in mathematics, especially in the fields of set theory, analysis, and topology.

In mathematical analysis, topology is built by metric spaces. A mathematician named Maurice Frechet first introduced the concept of metric spaces in 1906. Metric is a concept used to define the distance between elements in a space. A metric space is a set in which a metric axiom is satisfied. The set X in which the metric axiom d is satisfied is written as (X, d) and is called a metric space [2].

Many mathematicians have conducted research on the concept of metric spaces. In 2006, Mustafa and Sims in their journal entitled "*A New Approach to Generalized Metric Spaces*" introduced the concept of G −metric space which is a generalization or extension of the concept of metric space. From that metric space, a definition for G −metric space can be produced and its topology is introduced [3]. Research on G −metric space continues to grow until now, for example in [4]-[8], and many more.

The discussion of compact sets has been in Hausdorff space [9], topological space [10], metric space [11], and fuzzy metric space [12]. Applications of compact sets have been widely used, usually related to topology and mathematical analysis. The discussion of the compactness of G −metric space has been discussed before, for example in [13] which discusses the finite nature of Bourbaki−G and compact local uniform−G and [14] which discusses the fixed point theorem on compact G −metric space. Therefore, to expand the discussion on the concept of compactness of G −metric spaces, this research paper will discuss the proof of theorems related to the properties of compact sets in G −metric spaces.

2. Literature Survey

A. G −metric Spaces

Definition 2.1. [3] *Given a nonempty set X, $G: X \times X \times X \longrightarrow \mathbb{R}^+$, and $x, y, z, a \in X$ which if satisfied:*

(G1) $G(x, y, z) = 0$ *if and only if* $x = y = z$
(G2) $0 < G(x, x, y)$ *with* $x \neq y$
(G3) $G(x, x, y) \leq G(x, y, z)$ *with* $y \neq z$
(G4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = G(y, z, x) = G(z, x, y) = G(z, y, x)$ *(symmetrical across all three variables)*
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *(rectangular inequality)*

then the function G is called the G-metric on X and (X, G) is called a G-metric space

Theorem 2.1 [3] *Given (X, G) is a G-metric space. Then for any $x, y, z, a \in X$ holds:*

(1) *If* $G(x, y, z) = 0$ *then* $x = y = z$
(2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$
(3) $G(x, y, y) \leq 2G(y, x, x)$
(4) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$
(5) $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$
(6) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$
(7) $|G(x, y, z) - G(x, y, a)| \leq \max\{G(a, z, z), G(a, a, z)\}$
(8) $|G(x, y, z) - G(x, y, a)| \leq G(x, a, z)$
(9) $|G(x, y, z) - G(y, z, z)| \leq \max\{G(x, z, z), G(x, x, z)\}$

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(10) $|G(x,y,y)-G(x,x,y)| \leq \max\{G(x,x,y), G(x,y,y)\}$

Theorem 2.2. [3] *Given (X,G) is a G-metric space. Then we can define the metric d which is constructed by the G-metric. For all $x,y \in X$ holds $d_G(x,y) = G(x,y,y) + G(x,x,y)$.*

Definition 2.2. [3] *Given (X,G) is a G-metric space. For every $p \in X$, $r \in \mathbb{R}$, and $r > 0$, the boundary with center p and radius r is $B_G(p,r) = \{y \in X: G(p,y,y) < r\}$.*

Theorem 2.3. [3] *Given (X,G) is a G-metric space. For every $p \in X$, $r \in \mathbb{R}$, $r > 0$, it holds:*

(1) *If $G(p,x,y) < r$ then $x,y \in B_G(p,r)$*
(2) *If $y \in B_G(p,r)$, then there exists $\delta > 0$ such that $B_G(y,\delta) \subseteq B_G(p,r)$*

Theorem 2.4. [3] *Given (X,G) is a G-metric space and $B_{d_G}(p,r)$ is the neighborhood of the metric constructed by the G-metric. Then for every $p \in X$, $r \in \mathbb{R}$, and $r > 0$, we can obtain $B_G\left(p,\frac{1}{3}r\right) \subseteq B_{d_G}(p,r) \subseteq B_G(p,r)$.*

Based on the above Theorem, it follows that the G −metric topology, $\tau(G)$, is equivalent to the metric topology constructed from d_G. Thus, G −metric spaces are topologically equivalent to metric spaces.

Definition 2.3. [15] *Given (X,G) is a G-metric space and $E \subset X$. The point $p \in X$ is called a limit point of E if for every neighborhood of the point p with radius r, $r \in \mathbb{R}$, $r > 0$, that is $B(p,r)$ holds*

$$(B(p,r) - \{p\}) \cap E \neq \emptyset$$

B. Open Set

Definition 2.4. [15] *Given (X,G) is a G-metric space and $E \subset X$. The set E is said to be open if for every $p \in E$ there is $r \in \mathbb{R}$, $r > 0$, such that $B_G(p,r) \subset E$.*

C. G-convergent, G-Cauchy, and G-complete

Definition 2.5. [16] *Given (X,G) is a G-metric space. The sequence (x_n) on X is said to be G-convergent to x if $lim\big(G(x,x_n,x_m)\big) = 0$. That is, for every $r \in \mathbb{R}$, $r > 0$, there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < r$ for all $n,m \in \mathbb{N}$ with $n,m \geq N$.*

Definition 2.6. [3] *Given (X,G) is a G-metric space. The sequence (x_n) on X is said to be G-Cauchy if for every $r \in \mathbb{R}$, $r > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < r$ for all $n,m,l \geq N$.*

Definition 2.7. [3] *Given (X,G) is a G-metric space. X is said to be G-complete if for every G-Cauchy sequence on X is a G-convergent sequence on X.*

Theorem 2.5. [3] *Given (X,G) is a G-metric space and $\{F_n: n \in \mathbb{N}\}$ is a collection of nonempty closed subsets on X with $F_{n+1} \subset F_n$. Then X is G-complete if and only if*

$$\bigcap_{n\in\mathbb{N}} F_n = \{x\}, x \in X$$

Definition 2.8. [3] *Given (X,G) is a G-metric space, $E \subset X$ and $r \in \mathbb{R}$, $r > 0$. E is called r-net of X if for any $x \in X$, there exists a point $p \in E$ such that $x \in B_G(p,r)$. If E is finite, then E is called finite r-net.*

D. G-sequentially Compact

Definition 2.9. *[3] Given (X,G) is a G-metric space. The set $E \subset X$ is said to be G-sequentially compact if every sequence on E has a G-convergent subsequence to a point in E. (X,G) is said to be G-sequentially compact if X is G-sequentially compact.*

3. Main Result

Before we study the properties of compact sets in G −metric spaces, we first define compact set. In this section, the G −metric space is denoted as (X,G), unless stated otherwise.

Definition 3.1. [17] *Given (X,G) is a G-metric space and $K \subset X$, and I* is the indexed set. *The collection of open sets $\mathcal{G} = \{G_a \subset X: a \in I\}$ is called the open cover of K if*

$$K \subset \bigcup_{G_a\in\mathcal{G}} G_a = \bigcup_{a\in I} G_a$$

Example 3.1. In a G-metric space, $(\mathbb{R},G)$ defined as $G(x,y,z) = |x-y| + |x-z| + |y-z|$ for each $x,y,z \in \mathbb{R}$, the collections $\mathcal{G} = \left\{\left(-\frac{p}{2},\frac{p}{2}\right): p \in [0,\infty)\right\}$, and $\mathcal{G}^* = \{\mathbb{R}\}$ are an open cover for $A = [0,\infty)$, respectively.

Example 3.2. In any G −metric space, (X,G) and any $K \subset X$ with $K \neq \emptyset$, then $\mathcal{G} = \{B_G(p,r): p \in K\}$ is open cover of K.

Definition 3.2. [18] *Given (X,G) is a G-metric space. If $\mathcal{G}$ $\mathcal{G}$ is an open cover of $K \subset X$ and $\mathcal{G}' \subset \mathcal{G}$ with $\mathcal{G}' = \{G_1, G_2, \dots, G_n\}$ such that*

$$K \subset \bigcup_{G_i\in\mathcal{G}'} G_i = \bigcup_{i=1}^{n} G_i$$

then $\mathcal{G}'$ is called a finite subcover of $\mathcal{G}$ for K

Definition 3.3. [19] *Given (X,G) is a G-metric space. A set $K \subset X$ is compact if for every open cover of K has finite subcovers.*

Example 3.3. In a real G −metric space, with

$$G(x,y,z) = \begin{cases} 0, \text{for } x = y = z \\ 5, \text{for others} \end{cases}$$

then set $A = [0,1]$ is not a compact set because there exists an open cover $\mathcal{G} = \{\{p\}: p \in [0,1]\}$ for A with the condition that every finite subcover of $\mathcal{G}$ is not a cover of A.

Example 3.4. In any G −metric space (X, G), the finite set $K = \{x_1, x_2, \dots, x_n\}$ is compact set. For any open cover $\mathcal{G} = \{G_a\}$ for K and for any $x_i \in K$ can be found $G_{x_i} \in \mathcal{G}$ with $x_i \in G_{x_i}$. Let $\mathcal{G}' = \{G_{x_1}, G_{x_2}, \dots, G_{x_n}\}$. Then we get $\mathcal{G}' \subset \mathcal{G}$ with the property

$$K \subset \bigcup_{G_{x_i} \in \mathcal{G}'} G_{x_i}$$

So, it is proved that K is a compact set.

Some properties of compact sets on *G*-metric spaces are given in the following theorems. Theorem 3.1 says that a closed set which is a subset of a compact set is a compact set.

Theorem 3.1. *In a G-metric space, if $K \subset X$ is a compact set, $E \subset K$, and E is a closed set, then E is a compact set.*
Proof: Let $\mathcal{G}$ be an open cover of E. Since E is a closed set, then E^C is an open set. Since $E \subset K$ and $\mathcal{G}$ are open cover of E then $\mathcal{G}^* = \mathcal{G} \cup \{E^C\}$ is open cover for K. Since K is a compact set, that $G_1, G_2, G_3, \dots, G_n \in \mathcal{G}^*$ such that

$$E \subset K \subset \bigcup_{i=1}^{n} G_i$$

Note that $\{G_1, G_2, G_3, \dots, G_n\} - \{E^C\} \subset \mathcal{G}$ and is a cover of E. So, it is proved that E is compact.

Furthermore, it will be proved that the compact set in the *G*-metric space is a closed and bounded set.

Theorem 3.2. *In a G-metric space, if $K \subset X$, K is a compact set, then K is closed and bounded.*
Proof: (i) Let $q \in K^C$ or $q \notin K$. Consequently, $p \neq q$ for all $p \in K$. Based on definition, for all $p \in K$ holds $G(p,p,q) > 0$. Then, it can be made $B_G(p,r)$ and $B_G(q,r)$ which are the neighborhood with radius r_p with $0 < r_p < \frac{1}{2}G(p,p,q)$ and center p and q. Therefore, it can be obtained that $B_G(p,r_p) \cap B_G(q,r_p) = \emptyset$. Based on definition, $\{B_G(p,r_p): p \in K\}$ is an open cover of K. Since K is a compact set, then there exists $p_1, p_2, \dots, p_n \in K$ such that

$$K \subset \bigcup_{i=1}^{n} B_G(p_i, r_i)$$

Will be selected $r = \min\{r_i: 1 \le i \le n\} > 0$. Then $q \in B_G(q,r) \subset B_G(q,r_i)$ with $i = 1,2,\dots,n$. Furthermore, it can be obtained

$$\begin{aligned}(B_G(q,r) \cap K) &\subset \left(B_G(q,r) \cap \left(\bigcup_{i=1}^{n} B_G(p_i,r_i)\right)\right) \\ &= \bigcup_{i=1}^{n} \big(B_G(q,r) \cap B_G(p_i,r_i)\big) \\ &\subset \bigcup_{i=1}^{n} \big(B_G(q,r_i) \cap B_G(p_i,r_i)\big) = \bigcup_{i=1}^{n} \emptyset \\ &= \emptyset\end{aligned}$$

So $B_G(q,r) \cap K = \emptyset$ which results in $B_G(q,r) \subset K^C$. Hence, if $q \in K^C$, then there exists $r \in \mathbb{R}$, $r > 0$ such that $B_G(q,r) \subset K^C$. In other terms, q is an interior point K^C or K^C is an open set. So K is a closed set. (ii) Based on definition, for all $p \in K$ can be built $B_G(p,1)$ which means the neighborhood with center p and radius 1 and $\{B_G(p,1): p \in K\}$ is a open cover of K. Since K is a compact set, then there exists $p_1, p_2, \dots, p_n \in K$ such that,

$$K \subset \bigcup_{i=1}^{n} B_G(p_i, 1).$$

Let $q \in X$, $q = p_1$ as a fixed point, and

$$M - 1 = \text{maks}\{G(q,p_2,p_2), G(q,p_3,p_3), \dots, G(q,p_n,p_n)\}.$$

For any $s \in K$, there exists $p_m \in K$ with $1 \le m \le n$ such that $s \in B_G(p_m, 1)$. This means, $G(s,s,p_m) < 1$. Based on definition, we can obtain

$$G(q,s,s) \le G(q,p_m,p_m) + G(p_m,s,s)$$
$$G(q,s,s) \le (M-1) + 1$$
$$G(q,s,s) \le M,$$

for each $s \in K$. This shows that K bounded.

By Theorem 3.2, in a *G*-metric space, every compact set is a closed and bounded set, but a closed and bounded set in a *G*-metric space is not necessarily a compact set. For example, consider the following example.

Example 3.5. In a real G −mtric space with,

$$G(x,y,z) = \begin{cases} 0, \text{for } x = y = z \\ 5, \text{for others} \end{cases}$$

then: (i) Set $A = [0,1]$ is closed, because in this metric space, every subset of $\mathbb{R}$ is an open set and also a closed set.
(ii) The diameter of A,

$$diam([0,1]) = Sup\{G(x,y,z): x,y,z \in [0,1]\} = 5 < \infty.$$

So $A = [0,1]$ is a bounded set.
By Example 3.3, $A = [0,1]$ is not compact set.

Theorem 3.3. *Given an indexed set I. If $\{K_a: a \in I\}$ is a collection of compact sets on the G −metric space, such that each of its infinite subcollections has a non-empty intersection, then*

$$\bigcap_{a\in I} K_a \neq \emptyset$$

Proof: In the theorem above, it means that if K_a is a compact set and

$$\bigcap_{i=1}^{n} K_{a_i} \neq \emptyset$$

for every $n \in \mathbb{N}$, then

$$\bigcap_{a} K_a \neq \emptyset$$

Take any member of the collection suppose K_{a_0}. Since K_a is a compact set, by Theorem 5.3, then K_a is closed. Since K_a is closed, then ${K_a}^C$ is open. Suppose ${K_a}^C = G_a$, then G_a is open. We will show that

$$K_{a_0} \cap \left(\bigcap_{a\neq a_0} K_a \right) \neq \emptyset$$

Will be proven by proof of contradiction. Suppose

$$K_{a_0} \cap \left(\bigcap_{a\neq a_0} K_a \right) = \emptyset$$

then

$$K_{a_0} \subset \left(\bigcap_{a\neq a_0} K_a \right)^C = \bigcup_{a\neq a_0} {K_a}^C = \bigcup_{a\neq a_0} G_a$$

$$K_{a_0} \subset \bigcup_{a\neq a_0} G_a$$

So, $\{G_a\}_{a\neq a_0}$ is an open cover for K_{a_0}. Since K_{a_0} is compact, then there exists $a_1, a_2, a_3, \ldots, a_n$ such that

$$K_{a_0} \subset \bigcup_{i=1}^{n} G_{a_i} = \bigcup_{i=1}^{n} K^C{}_{a_i} = \left(\bigcap_{i=1}^{n} K_{a_i} \right)^C$$

This shows that

$$K_{a_0} \subset \left(\bigcap_{i=1}^{n} K_{a_i} \right)^C$$

which resulted in

$$K_{a_0} \cap \left(\bigcap_{i=1}^{n} K_{a_i} \right) = \emptyset$$

or $K_{a_0} \cap K_{a_1} \cap K_{a_2} \cap \ldots \cap K_{a_n} = \emptyset$. However, it contradicts the statement that every finite subcollection has a non-empty intersection. Thus, the supposition is false and should be

$$K_{a_0} \cap \left(\bigcap_{a\neq a_0} K_a \right) \neq \emptyset$$

Since K_{a_0} is arbitrary, then

$$\bigcap_{a} K_a \neq \emptyset$$

Thus, Theorem 3.3 is proven.

Corollary 3.1. *If* $\{K_n\}_{n\geq 1}$ *is a non-empty collection of compact sets with* $K_{n+1} \subset K_n$ *for every* $n \in \mathbb{N}$*, then*

$$\bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$$

Theorem 3.4. *Given* (X, G) *is a G-metric space,* $K \subset X$*, and* K *is compact. If* E *is an infinite subset of* K *then* E *has a limit point at* K.

Proof: We will be proven by proof of contradiction. Let E has no limit point on K. This means that if $p \in K$ then p is not a limit point of E. Based on definition, $B_G(p,r) - \{p\} \cap E = \emptyset$. Thus, each $B_G(p,r)$ contains only the point p itself which is contained in E. Thus, by definition, since $E \subset K$ then $\{\mathrm{B_G(p,r)}: \mathrm{p} \in \mathrm{K}\}$ is an open cover of K and E.

$$E \subset K \subset \bigcup_{p\in K} B_G(p,r)$$

Since K is compact set, then by definition, there exists $p_1, p_2, p_3, \ldots, p_n \in K$ such that

$$K \subset \bigcup_{i=1}^{n} B_G(p_i, r_i)$$

Also applies to E because $E \subset K$, that is

$$E \subset \bigcup_{i=1}^{n} B_G(p_i, r_i)$$

Meanwhile, each $B_G(p_i, r_i)$ contains only one point in E. This means that the members of E are finite. This contradicts the known statement that E is an infinite subset, so the supposition is false. Therefore, E must have a limit point on K.

Theorem 3.5. *Given* (X, G) *is a G-metric space,* (Y, G) *is a G-metric subspace, and* $K \subset Y$. K *is compact relative to* Y *if and only if* K *is compact relative to* X.

Proof:

(1) Will be proved that if K is compact relative to Y then K is compact relative to X.

Suppose $\{B_G(p,r): p \in K\}$ is an open cover of K on X. Based on theorem and definition, $\{B_G(q,r): q \in K\}$ with $B_G(q,r) = B_G(p,r) \cap Y$ is an open cover of K against Y. Then, based on definition, there exists $q_1, q_2, \dots, q_n$ such that,

$$K \subset \bigcup_{i=1}^{n} B_G(q_i, r_i) \subset \bigcup_{q\in K} B_G(q,r)$$

That cover K, because K is compact relative to Y.

Therefore, there exists $p_1, p_2, \dots, p_n$ such that,

$$K \subset \bigcup_{i=1}^{n} B_G(p_i, r_i) \subset \bigcup_{p\in K} B_G(p,r)$$

that cover K. Thus, K is compact relative to X.

(2) Will be proved that if K is compact relative to X then K is compact relative to Y.
Suppose $\{B_G(q,r): q \in K\}$ is an open cover of K against Y. Based on theorem, $B_G(q,r) = B_G(p,r) \cap Y$ with $B_G(p,r)$ is an open set relative to X, so that $\{B_G(p,r): p \in K\}$ is an open cover of K relative to X. Since K is compact relative to X, then by definition, $\{B_G(p,r): p \in K\}$ contains a finite subcovers that cover K.

$$K \subset \bigcup_{i=1}^{n} B_G(p_i, r_i) \subset \bigcup_{p\in K} B_G(p,r)$$

Since $B_G(q,r) = B_G(p,r) \cap Y$, then $\{B_G(q,r): q \in K\}$ contains a finite subcovers which is a cover of K as well. Thus, K is compact relative to Y.

Corollary 3.2. *Given (X,G) is a G-metric space, (Y,G) is a G-metric subspace, and $K \subset Y$. If $Y = X$, then K is not compact relative to Y.*

Definition 3.4. [16] *Given (X,G) is a G-metric space. The sequence (x_n) on X is said to be G-convergent to x if $lim\big(G(x,x_n,x_m)\big) = 0$. That is, for every $r \in \mathbb{R}$, $r > 0$, there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < r$ for all $n,m \in \mathbb{N}$ with $n,m \geq N$.*

Example 3.6. Let $(\mathbb{R}, G)$ is a *G*-metric space, with

$$G(x,y,z) = |x-y| + |x-z| + |y-z|$$

for all $x,y,z \in \mathbb{R}$. Sequence $(x_n) = \left(\frac{1}{n+1}\right)$ is a *G*-convergent to $x = 0$, since for every $r \in \mathbb{R}$, $r > 0$ there exists $N \in \mathbb{N}$, i.e., $N =$ the smallest natural number greater than $\frac{4}{r}$ such that $\frac{4}{N} < r$. For all $n,m \in \mathbb{N}$ with $n,m \geq N$, we have

$$G(x,x_n,x_m) = |x-x_n| + |x-x_m| + |x_n-x_m|$$

$$= \left|0-\frac{1}{n+1}\right| + \left|0-\frac{1}{m+1}\right| + \left|\frac{1}{n+1}-\frac{1}{m+1}\right|$$

$$\leq \frac{1}{n+1}+\frac{1}{m+1}+\frac{1}{n+1}+\frac{1}{m+1}$$

$$\leq \frac{2}{n+1}+\frac{2}{m+1} \leq \frac{2}{N+1}+\frac{2}{N+1} \leq \frac{2}{N}+\frac{2}{N} \leq \frac{4}{N} < r.$$

Definition 3.5. [3] *Given (X,G) is a G-metric space. The sequence (x_n) on X is said to be G-Cauchy sequence if for every $r \in \mathbb{R}$, $r > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < r$ for all $n,m,l \geq N$.*

Example 3.7. Let $(\mathbb{R}, G)$ is a *G*-metric space, with

$$G(x,y,z) = |x-y| + |x-z| + |y-z|$$

for all $x,y,z \in \mathbb{R}$.
A sequence $(x_n) = \left(\frac{1}{n}\right)$ is a *G*-Cauchy sequence because for each $r \in \mathbb{R}$, $r > 0$, there exists $N \in \mathbb{N}$ which is the smallest natural number greater than $\frac{6}{r}$ such that $\frac{6}{N} < r$. By Definition 3.5, for all $n,m,l \geq N$ holds
$G(x_n,x_m,x_l) = |x_n-x_m| + |x_n-x_l| + |x_m-x_l| = \left|\frac{1}{n}-\frac{1}{m}\right| + \left|\frac{1}{n}-\frac{1}{l}\right| + \left|\frac{1}{m}-\frac{1}{l}\right| \leq \frac{1}{n}+\frac{1}{m}+\frac{1}{n}+\frac{1}{l}+\frac{1}{m}+\frac{1}{l} = \frac{2}{n}+\frac{2}{m}+\frac{2}{l} \leq \frac{2}{N}+\frac{2}{N}+\frac{2}{N} \leq \frac{6}{N} < r$
Thus, it is proven that the sequence $(x_n) = \left(\frac{1}{n}\right)$ is a *G*-Cauchy sequence.

Definition 3.6. [3] *Let (X,G) is a G-metric space. X is said to be G-complete if for every G-Cauchy sequence on X is a G-convergent sequence on X.*

The definition of the *net* of a set has been given by [3] which will be used as a discussion of compact sets from another point of view.

Definition 3.7. [3] *Let (X,G) is a G-metric space, $E \subset X$ and $r \in \mathbb{R}$, $r > 0$. E is called r-net of X if for any $x \in X$, there exists a point $p \in E$ such that $x \in B_G(p,r)$. If E is finite, then E is called finite r-net of X.*

Example 3.8. In the real G-metric space, which is defined as*:*

$$G(x,y,z) = |x-y| + |x-z| + |y-z|$$

for every $x,y,z \in \mathbb{R}$ then

(a) $\mathbb{Q}$ is $1 - net$ to $\mathbb{R}$
(b) $\mathbb{Z}$, the set of all integers, is $3 - net$ for $\mathbb{R}$.

Example 3.9. In real $G - metric$ which

$$G(x,y,z) = \begin{cases} 0, \text{for } x = y = z \\ 5, \text{for others} \end{cases}$$

then

$$B_G(p,k) = \{y \in \mathbb{R}: G(p,y,y) < k\} = \begin{cases} \{p\}, \text{for } k \leq 5 \\ \mathbb{R}, \text{for } k > 5 \end{cases}$$

Therefore:

(a) $A = \{3\}$ is $7 - net$ for $\mathbb{R}$
(b) $A = \{3\}$ is not a $4 - net$ for $\mathbb{R}$
(c) If r is an arbitrary real number with $0 < r \leq 5$, then every finite subset of $\mathbb{R}$ which is not empty is not a $r - net$ for $\mathbb{R}$.

Definition 3.8. [3] *Let (X, G) is a G-metric space. The set $A \subset X$ is said to be G-totally bounded if for every $r \in \mathbb{R}$, $r > 0$, there exists a finite r-net of A.*

In other words, the set A is said to be G-totally bounded if for every $r \in \mathbb{R}$, $r > 0$, there exists $x_1, x_2, x_3, , ..., x_n \in A$ such that $\mathcal{G} = \{B_G(x_i, r): i = 1, 2, 3, \ldots, n\}$ is an open cover of A or

$$A \subset \bigcup_{1 \leq i \leq n} B_G(x_i, r)$$

Example 3.10. Let $(\mathbb{R}, G)$ is a G-metric space *with* $G(x, y, z) = |x - y| + |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ then:
(i) $\mathbb{N}$ is not a G-totally bounded set
(ii) $A = [0,1]$ is a G-totally bounded set
Proof. For any $p \in \mathbb{R}$, and $r \in \mathbb{R}$ with $r > 0$ then

$$\begin{aligned} B_G(p, r) &= \{x \in \mathbb{R}: G(p, x, x) < r\} \\ &= \{x \in X: |p - x| + |p - x| + |x - x| < r\} \\ &= \left(p - \frac{r}{2},\ p + \frac{r}{2}\right) \end{aligned}$$

(i) Since there is $r \in \mathbb{R}$ with $r > 0$, e.g $r = 1$, so that for every finite collection $\mathcal{G} = \{B_G(p_i, 1): p_i \in \mathbb{N}, i = 1, 2, 3, \ldots n\}$ is not a cover for $\mathbb{N}$, then $\mathbb{N}$ is not a G-totally bounded set.
(ii) For any $r \in \mathbb{R}$ with $r > 0$, there exists $n \in \mathbb{N}$ with $\frac{1}{n} < \frac{r}{2}$. Consequently, $\mathcal{G} = \left\{B_G\left(\frac{i-1}{n}, r\right): i = 1, 2, 3, \ldots n\right\}$ is an open cover for $A = [0,1]$. So, $A = [0,1]$ is a G-totally bounded set.

Example 3.11. Let $(\mathbb{R}, G)$ is a G-metric space with

$$G(x, y, z) = \begin{cases} 0, \text{for } x = y = z \\ 5, \text{for others} \end{cases}$$

for all $x, y, z \in \mathbb{R}$. The set $A = [0,1]$ is not a G-totally bounded set, because there exists $r \in \mathbb{R}$ with $r > 0$, e.g. $r = 2$ so that for every finite collection $\mathcal{G} = \{B_G(p_i, 2): p_i \in [0, 1], i = 1, 2, 3, \ldots n\} = \{\{p_i\} \subset [0, 1]:\ i = 1, 2, 3, \ldots n\}$ is not a cover for $[0, 1]$.

Definition 3.9. [3] *Let (X, G) is a G-metric space. The set $E \subset X$ is said to be G-sequentially compact if every sequence on E has a G-convergent subsequence to a point in E.*

Example 3.12. Let $(\mathbb{R}, G)$ is a G-metric space with $G(x, y, z) = |x - y| + |x - z| + |y - z|$. The set $E = [0,1]$ is a G-sequentially compact set.
Proof. Since in the G-metric space is a closed and bounded set, then every sequence on $E = [0,1]$ there is always exist a subsequence on $E = [0,1]$ that is G-convergent to a point in E.

Example 3.13. Let $(\mathbb{R}, G)$ is a G-metric space with

$$G(x, y, z) = \begin{cases} 0, \text{for } x = y = z \\ 5, \text{for others} \end{cases}$$

for all $x, y, z \in \mathbb{R}$. The set $A = [0,1]$ is not a G-sequentially compact set, because there is a sequence $(x_n) = \left(\frac{1}{n}\right)$ on E such that any subsequence of it is not G-convergent to a point in E.

In a metric space with usual metric on $\mathbb{R}$, every closed and bounded set is a compact set. However, this is different from the G-metric space. In the G-metric space, there is a closed and bounded set, but the set is not a compact set on the G-metric space (Example 3.5). Since there is a set in the G −metric space which is closed and bounded but not compact, then we will discuss the necessary and sufficient conditions for a set in the G −metric space to be compact.

The following theorem is a necessary and sufficient conditions for a set in a G-metric space is a compact set.

Theorem 3.6. *Let (X, G) is a G-metric space and $K \subset X$. Then the following three statements are equivalent.*
(1) *K is a G-complete and G-totally bounded set*
(2) *K is a compact set*
(3) *K is a G-sequentially compact set*
Proof: (1)→(2): (If K is a complete set-G and totally finite-G, then K is a compact set).
Contradiction proof is used. Suppose K is not compact, and suppose $\mathcal{G}$ is an open cover for K which does not contain a finite subcover for K. Suppose $\mathcal{G}' = \{G_1, G_2, G_3 \ldots, G_m\} \subset \mathcal{G}$ with $K \not\subset \bigcup_{G_i \in \mathcal{G}'} G_i$. So there exists $P \subset K$ with $P \not\subset \bigcup_{G_i \in \mathcal{G}'} G_i$. Since K G-totally bounded, then for any $r \in \mathbb{R}, r > 0$ can be chosen $p_1, p_2, p_3, \ldots, p_{n_r} \in K$ so that $K \subset \bigcup_{j=1}^{n_r} B_G(p_j, r)$.
Formed the set

$$H_{i,j} = G_i \cap B_G\left(p_j, r\right),$$
$$i = 1, 2, 3, \ldots, m \text{ and } j = 1, 2, 3, \ldots, n_r.$$

Then $H_{i,j}$ is an open set with the property that

$$K \not\subset \bigcup_{1 \leq i \leq m; 1 \leq j \leq n_r} H_{i,j}.$$

If for every natural number k taken $r = \frac{1}{k}$, and $P_k \subset K$ with $P \not\subset \bigcup_{1 \leq i \leq m; 1 \leq j \leq n_r} H_{i,j}$, F_k is the closure of P_k, then obtained:
(i) F_k is a closed set
(ii) $F_k \subset K$
(iii) $F_k \subset F_{k+1}$
(iv) $F_k \neq \emptyset$
Since K is a compact set, it follows from (i) and (ii) that F_k is a compact set. Consequently,

$$\bigcap_{k \in \mathbb{N}} F_k \neq \emptyset$$

For example, $p_0 \in \bigcap_{k \in \mathbb{N}} F_k$ and since K is a G-complete set, then there exists $G_i \in \mathcal{G}$ so that $p_0 \in G_i$ and $p_0 \in P$. From the other side, $P \not\subset \bigcup_{G_i \in \mathcal{G}'} G_i$. So, there is a contradiction. So, the correct one is K is a compact set.

(2)→(3) (If K is compact, then K is G-sequentially compact). Suppose (p_n) is a sequence on K. Let F_n be a closure of the nonempty set $\{p_k : k > n\}$. Then (F_n) is a monotonically decreasing sequence of the nonempty closed sets. As a result, there exists $p_0 \in K$ so that

$$p_0 \in \bigcap_{n \in \mathbb{N}} F_n$$

For every n, we have:

(i) p_0 is contained in the closure of $\{p_k : k > n\}$,

(ii) The neighborhood of $B_G\left(p_0, \frac{1}{k}\right)$ has a nonempty intersection with $\{p_k : k > n\}$,

then we can choose (n_k) such that for each index k, $G(p_0, p_{n_k}, p_{n_k}) < \frac{1}{k}$. So, the subsequence (p_{n_k}) is G-convergent to p_0. Thus, K is G-sequentially compact set.

(3)→(1) (If K is G-sequentially compact set, then K is G-complete and G-totally bounded).

Suppose K is not G-totally bounded. Then, there exists a real number $r > 0$ such that for any finite set $H \subset K$ with $K \not\subset \bigcup_{p \in H} B_G(p, r)$. Selected $p_1 \in K$ with $p_1 \notin B_G(p, r)$. Selected $p_2 \in K$ with $G(p_1, p_2, p_2) > r$. Since $K \not\subset B_G(p_1, r) \cup B_G(p_2, r)$, then can be selected $p_3 \in K$ with $G(p_1, p_3, p_3) > r$ and $G(p_2, p_3, p_3) > r$. Furthermore, with the same process, we can obtain the sequence (p_n) at K with $G(p_k, p_n, p_n) > r$ for $n > k$. So, the sequence (p_n) cannot have a subsequence that G-convergent. Consequently, K is not a G-sequentially compact set. Contradiction to what is known, i.e. K is sequentially compact. So, the correct one is K is a G-totally bounded set.

To show that K is a G-complete, suppose (x_n) is a G-Cauchy sequence on K. Since K is G-sequentially compact, then the subsequence of (x_n) G-convergent to $p \in K$. So, all sequences G-convergent to p. Consequently, (x_n) G-convergent to $p \in K$ or K is a G-complete set.

Example 3.15. On a real G −metric, which $G(x, y, z) = |x - y| + |x - z| + |y - z|$. $K = [0,1]$ is a compact set

Proof: First, will be shown that K is a G -complete set. Based on Definition 3.6., K is said to be G-complete if for every G -Cauchy sequence on K is a G-convergent sequence to a point in K. Suppose (x_n) is a G-Cauchy sequence on $[0,1]$. Since $[0,1]$ is a closed and bounded interval, then every G-Cauchy sequence (x_n) on $[0,1]$ will G-convergent to $x \in [0,1]$. Thus, K is a G-complete set.

Second, will be shown that K is a G-totally bounded set. Suppose $\mathcal{G} = \{B_G(p, r) | p \in K\}$ is the open cover of K. By definition, the neighborhood with center $p \in K$ and radius $r \in \mathbb{R}, r > 0$, that is

$B_G(p, r) = \{y \in K : G(p, y, y) < r\}$

$B_G(p, r) = \{y \in K : |p - y| + |p - y| + |y - y| < r\}$

$= \{y \in K : 2|p - y| < r\}$

$= \{y \in K : 2p - r < 2y < 2p + r\}$

$= \left\{y \in K : p - \frac{r}{2} < y < p + \frac{r}{2}\right\}$

$= \left(p - \frac{r}{2}, p + \frac{r}{2}\right)$

Based on Definitions 3.7 and 3.8, the set of K is said to be G-totally bounded if for every $r \in \mathbb{R}$, $r > 0$, there exists $\mathcal{G}' \subset \mathcal{G}$ with $\mathcal{G}' = \{B_G(x_1, r), B_G(x_2, r), \ldots, B_G(x_n, r)\}$ such that $\mathcal{G} = \{B_G(x, r) | x \in X\}$ is an open cover for K. Let $r \in \mathbb{R}, r > 0$. By Archimedian property, we can be chosen $n \in \mathbb{N}$ such that $r < \frac{1}{n}$ and $K = [0,1]$ can be divided into as many n equally spaced subintervals, i.e., each is spaced $\frac{1}{n}$. For each subinterval, one of the rational endpoints of each subinterval can be chosen as the point at $B_G(p, r)$. This ensures that every point in $K = [0,1]$ is in the neighborhood whose distance from point p, less than r. Suppose y is an arbitrary point in K. Then point y must be contained in one of the subintervals. If we take y as the rational endpoint of the subinterval, then $y \in B_G(p, r)$. Since $K = [0,1]$ can be divided into finitely many equally spaced subintervals such that

$$K \subset \bigcup_{p \in K} B_G(p, r)$$

then K is G-totally bounded set.

Example 3.11. Let $(\mathbb{R}, G)$ is a G-metric space with

$$G(x, y, z) = \begin{cases} 0, \text{for } x = y = z \\ 5, \text{for others} \end{cases}$$

for all $x, y, z \in \mathbb{R}$. By Example 3.11, the set $A = [0,1]$ is not a G-totally bounded. So, by Theorem 3.6., the set $A = [0,1]$ is not compact.

4. Conclusion

In G-metric spaces, compact sets have closed and bounded properties. However, a closed and bounded set is not necessarily a compact set. The requirement that must be added in order for a set to be said to be compact in the G -metric space is that the set must be G-complete and G −totally finite.

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