International Journal of Research in Engineering, Science and Management
Volume 5, Issue 7, July 2022
https://www.ijresm.com | ISSN (Online): 2581-5792

Some Properties Regarding Cesaro Sequence Space of An Absolute Type

Arief Rachman Hakim[1], Manuharawati[2*], Muhammad Jakfar[3]
[1,2,3]*Department of Mathematics, Universitas Negeri Surabaya, Surabaya, Indonesia*

***Abstract*: The study of sequence spaces has been a great interest recently. A number of books have been published in this area over the last few years. In addition, the sequence space has also been widely applied to various fields. The Cesàro Sequence Space of an Absolute type also one of the sequence space that being studied recently. It was introduced by a mathematician named J.S. Shue back in 1970. In this particular article, we give the proof of Cesàro Sequence Space of an Absolute type, with norm defined by $||x|| = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{\frac{1}{p}}$ for any real number p that satisfy $1 \leq p < \infty$ and $||x||_{ces_\infty} = \sup\left\{\frac{1}{n}\sum_{k=1}^{n}|x_k|, n \in \mathbb{N}\right\}$ being a Banach space.. We also conduct the proof of Cesàro Sequence Space of an Absolute type is a BK-space, FK-space, having the AK-property and also a solid sequence space.**

***Keywords*: Cesaro Sequence Space of an Absolute Type, Cesaro Sequence Space, Absolute type of sequence space, Solid Sequence Space, BK-space, FK-space, AK-property, Banach Space, Sequence space.**

1. Introduction

Sequence space is a set or collection whose members are sequences where the sequence itself is a function that maps the set of all natural numbers $\mathbb{N}$ to a set $\mathbb{S}$ [1]. One of the sequence spaces that has been widely studied is Cesaro Sequence Space of an Absolute type, which is symbolized by ces_p. Cesaro Sequence Space of an Absolute type is the set of all sequences of real numbers (x_n) where the series $\sum_{n\in\mathbb{N}}\left(\sum_{i=1}^{n}\frac{|x_n|}{n}\right)^p < \infty$ ofr a real numbers p with $1 \leq p < \infty$. And for $p = \infty$, Cesaro Sequence Space of an Absolute type is defined as the collection of all sequences of real numbers such that the value of $\sum_{i=1}^{n}\frac{|x_n|}{n}$ is finite for all natural numbers n.

At first, Dutch Mathematical Society publicly announces a problem regarding the duals of Cesaro Sequence Space in 1968. Two years later, a mathematician named Shiue [2] successfully solves the problem through an article in 1970. Leibowits [3] and Jager [4] then also investigate some properties of Cesaro Sequence Space. In 1978, Ng and Lee [5] introduces the non-absolute type of Cesaro Sequence Space that denoted by X_p. Sequence Space X_p denotes the set off all real number sequences that satisfy $\sum_{n\in\mathbb{N}}\left|\sum_{i=1}^{n}\frac{x_n}{n}\right|^p < \infty$ for a real number p greater than 1 whilst for $p = \infty$ they defined X_∞ as a collection of sequences which the value of $\left|\sum_{i=1}^{n}\frac{x_n}{n}\right|$ is finite for any natural number n. After that, Lee [6] studied about the α-dual of ces_p and β-dual of X_p.

Other research regarding the structure of Cesaro Sequence Space have also been studied by another researchers. Such as geometry properties of ces_p by Saejung [7], topological and algebraic properties of Cesaro Sequence Space that defined by modulus function [8], and the structure of Cesaro Function Space that related with Cesaro Sequence Space, Copson Function Space, and Copson Sequence Space [9].

In addition, the implementation of sequence space in various fields have also been studied lately. For example, Talo & Başar [10] studied the sequences space using fuzzy numbers. Malkowsky *et al.* [11] studied the implementation of Cesaro Sequence Space in Crystallography and another space that related to Cesaro Sequence Space. Some application of sequence space in clustering have been studied as well [12]. Recently, Khan *et al.* [13] made intuitionistic fuzzy distance measure based on Cesaro Paranormed Sequence Space.

Because of the benefit of the sequence spaces, we need to do more observation regarding the properties of sequences space especially Cesaro Sequence Space of an Absolute type. In this particular article, the proof of Cesaro Sequence Space of an Absolute type being a solid or normal sequence space as well as being a BK-space satisfying AK-property are being discussed.

2. Literature Survey

In this paper, the notation $(X, ||\bullet||)$ will be used to denote a space X that equipped with norm function $||\bullet||$. If there is some ambiguity, we will denote the norm of a space X by $||x||_X$ for every x in X. In order to be called a norm space, the norm function in X should be non-negative, close under scalar multiplication, satisfy the triangle inequality, and equal to zero if and only if x is the zero element of X. Before going to the main result, we will introduce some definition of Cesaro Sequence Space of an Absolute type, Banach space, solid sequence space, BK-space and AK-property.

A. *Cesaro Sequence Space of an Absolute Type*

The definition of Cesaro Sequence Space of an Absolute type

*Corresponding author: manuharawati@ unesa.ac.id

will be stated in Definition 2.1.

Definition 2.1 [6] *Cesaro Sequence Space of an Absolute type, denoted by ces_p, is a set of sequence of real numbers (x_n) that satisfy*

$$ces_p = \left\{(x_n) : \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p < \infty\right\}$$

where p is a real numbers greater than or equal to 1. While for $p = \infty$ the definition of Cesaro Sequence Space of an Absolute type is

$$ces_\infty = \left\{(x_n): \frac{1}{n}\sum_{k=1}^{n}|x_k| < \infty, x_n \in \mathbb{R}\ \forall n \in \mathbb{N}\right\}$$

Both ces_p and ces_∞ are said to be sequence space of an absolute type because for every sequence of real number $(x_n) \in ces_p$ (or ces_∞), implies the sequence (x_n) also in ces_p (or ces_∞). To have a better understanding in Cesaro Sequence Space of an Absolute type, here are some example of the element.

Example 2.2 The sequence $X = \left(\frac{1}{n} - \frac{1}{n+1}\right)$ is an element of ces_∞ and ces_p for every real numbers p such that $1 < p < \infty$.

Proof:

Notice that

$$\frac{1}{n}\sum_{k=1}^{n}|x_k| = \frac{1}{n}\sum_{k=1}^{n}\left|\frac{1}{k} - \frac{1}{k+1}\right|$$
$$= \frac{1}{n}\left[1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1}\right]$$
$$= \frac{1}{n}\left[1 - \frac{1}{n+1}\right]$$
$$= \frac{1}{n}\left[\frac{n}{n+1}\right]$$
$$= \frac{1}{n+1}$$

We can see that for every real numbers p greater than 1, we have

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p = \sum_{n=1}^{\infty}\left(\frac{1}{n+1}\right)^{\mathrm{p}} < \infty$$

and for every n being a natural number, the value of $\frac{1}{n+1}$ is finite. Hence $X \in ces_\infty$ and $X \in ces_p$ for a real number p where $1 < p < \infty$.

Example 2.3 The sequence $X = (m, 0,0,0, \dots)$ is an element of ces_p for every real number $p \in (1, \infty)$ and also an element of ces_∞ for $m \in \mathbb{R}$.

Proof:

Suppose m is any real number. Then $\forall n \in \mathbb{N}$, we have

$$\frac{1}{n}\sum_{k=1}^{n}|x_k| = \frac{1}{n}[m] < \infty$$

Moreover,

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p = \sum_{n=1}^{\infty}\left(\frac{m}{n}\right)^p = \mathrm{m}^{\mathrm{p}}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^p < \infty$$

Therefore, X is an element of Cesaro Sequence Space of an Absolute type for any real numbers p that satisfy $1 < p < \infty$ and also for $p = \infty$

Example 2.4 The sequence $(0,0,0, \dots) \in ces_p$ for any real number p satisfy $1 \le p < \infty$ and for $p = \infty$.

Proof:

We can see that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p = \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|0|\right)^p = \sum_{n=1}^{\infty}(0)^{\mathrm{p}} = 0$$

And also for every n being a natural number, the value of $\frac{1}{n}\sum_{k=1}^{n}|x_k|$ is equal to 0. In consequence, $(0,0,0, \dots)$ is an element of ces_p whenever $p \in [1, \infty)$ and $(0,0,0, \dots) \in ces_\infty$.

Example 2.5 The constant sequence $(x_n) = (m, m, m, \dots)$ of a real numbers m where $m \neq 0$ is in ces_∞ but not an element of ces_p where p is any real numbers greater than or equal to 1.

Proof:

Notice that $\forall n \in \mathbb{N}$, the value of $\frac{1}{n}\sum_{k=1}^{n}|m| = \frac{1}{n}[mn] = m < \infty$. Hence $(x_n) = (m, m, m, \dots)$ for a non-zero real number m is an element of ces_p. On the other hand, we have

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^{\mathrm{p}}$$
$$= \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|m|\right)^{\mathrm{p}}$$
$$= \sum_{n=1}^{\infty}\left(\frac{1}{n}[mn]\right)^{\mathrm{p}}$$
$$= \sum_{n=1}^{\infty}\mathrm{m}^{\mathrm{p}}$$
$$= \infty$$

Therefore, (x_n) is not in ces_p for any real number p whenever $1 \le p < \infty$.

B. Some Definition of Properties in Sequence Space

Definition 2.6 [14] *A norm space $(X, ||\bullet||)$ is said to be a Banach space iff $(X, ||\bullet||)$ is complete, that is every Cauchy Sequence on X is convergent.*

Definition 2.7 [15] *Given any sequence space X. X is said to be a solid (normal) sequence space iff for every $(x_n) \in X$ implies $(\alpha_n x_n)$ also an element of X whenever (α_n) be a sequence of scalar with $|a_n| \le 1, \forall n \in \mathbb{N}$.*

Definition 2.8 [16],[17] *A sequence space X is said to be a BK-space (Banach Coordinate Space) iff X satisfy these following condition*

a. X is a Banach Space.

b. Function $p_n: X \to \mathbb{C}$ is continuous for every natural number n where $p_n(x) = x_n$ and $x = (x_n) \in X$.

Lemma 2.9 [18] *Every BK-space is an FK-space.*

Definition 2.10 [18] *Given any FK-Space X. If every sequence $x \in X$ have a unique representation $x = \sum_{k\in\mathbb{N}} x_k e_k^{(k)}$ where*

$e^{(k)}$ is a sequence which $e_n^{(k)}$ is equal to 1 if $n = k$ and equal to 0 whenever $n \neq k$. In other word, $\lim_{n\to\infty} \sum_{k=1}^{n} x_k e^{(k)} = x$ or X is having the Schauder Basis.

3. Main Result

Theorem 3.1 *Cesaro Sequence Space of an Absolut type ces_p for every real number p satisfying $1 \leq p < \infty$ equipped with norm function*

$$||X||_{ces_p} = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{\frac{1}{p}}$$

is a Banach space.

Proof:

a. Suppose $X = (x_n) \in ces_p$. Since $|x_k| \geq 0$ for every $k \in \mathbb{N}$, then we have

$$0 \leq \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p < \infty$$

$$0 \leq \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{\frac{1}{p}} < \infty$$

$$0 \leq ||X||_{ces_p} < \infty$$

b. ($\Rightarrow$) If $X = (0,0,0,\dots)$ then

$$||X||_{ces_p} = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}0\right)^p\right)^{\frac{1}{p}}$$

$$\Leftrightarrow \left(\sum_{n=1}^{\infty}0^p\right)^{\frac{1}{p}} = 0$$

($\Leftarrow$) Conversely, assume that $x \neq \mathbf{0}$ and $||x||_{ces_p} = 0$, then there exist $k_1 \in \mathbb{N}$ such that x_{k1} is the first non-zero term in x. Therefore

$$||x||_{ces_p} = \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p = \sum_{n=k_1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p > 0$$

which contradict the fact that $||x||_{ces_p} = 0$. Hence $||x||_{ces_p} = 0$ if and only if $x = \mathbf{0}$.

c. Given any real number α, then

$$||\alpha X||_{ces_p} = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|\alpha x_k|\right)^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|\alpha||x_k|\right)^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{n=1}^{\infty}|\alpha|^p\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{\frac{1}{p}}$$

$$= |\alpha|\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{\frac{1}{p}}$$

$$= |\alpha|||X||_{ces_p}$$

d. According to triangle inequality, we have $|x_k + y_k| < |x_k| + |y_k|$ for any real numbers x_k and y_k. Hence for $X, Y \in ces_p$ we have

$$||X+Y||_{ces_p} = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k + y_k|\right)^p\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}(|x_k| + |y_k|)\right)^p\right)^{\frac{1}{p}}$$

And by using Minkowski inequality in sum, we get

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k| + \frac{1}{n}\sum_{k=1}^{n}|y_k|\right)^p\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|y_k|\right)^p\right)^{\frac{1}{p}}$$

$$= ||X||_{ces_p} + ||Y||_{ces_p}$$

e. From a, b, c, and d we have that ces_p is a norm space. Given $X = (x_n)$ be a Cauchy sequence on ces_p where $x_n = \left(x_n^{(k)}\right)$. Then for every positive real number ε, there exist $N \in \mathbb{N}$ such that any natural numbers m and q greater than or equal to N, we have

$$||x_m - x_q||_{ces_p} = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\left|x_m^{(k)} - x_q^{(k)}\right|\right)^p\right)^{\frac{1}{p}} < \varepsilon$$

$$\Leftrightarrow \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\left|x_m^{(k)} - x_q^{(k)}\right|\right)^p < \varepsilon^p$$

That means for every $n \in \mathbb{N}$, we get

$$\left(\frac{1}{n}\sum_{k=1}^{n}\left|x_m^{(k)} - x_q^{(k)}\right|\right)^p < \varepsilon^p \Leftrightarrow \frac{1}{n}\sum_{k=1}^{n}\left|x_m^{(k)} - x_q^{(k)}\right| < \varepsilon$$

Consequently, the sequence $\left(x^{(n)}\right) = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \dots)$ is a Cauchy sequence for any natural number n. Remember that $\left(x^{(n)}\right)$ is a Cauchy sequence on $\mathbb{R}$. Since $\mathbb{R}$ is a Banach space, therefore (x_n) is convergent $\forall n \in \mathbb{N}$. Suppose that $x_q^{(n)} \to y_n$ as $q \to \infty$. Then we can construct $y = (y_1, y_2, y_3, \dots)$. As of $q \to \infty$, we have

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\left|x_m^{(k)} - x_q^{(k)}\right|\right)^p < \varepsilon^p$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\left|x_m^{(k)} - y_k\right|\right)^p < \varepsilon^p$$

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\left|x_m^{(k)} - y_k\right|\right)^p\right)^{\frac{1}{p}} < \varepsilon$$

$$||x_m - y||_{ces_p} < \varepsilon$$

We can conclude that x_m is convergent to the sequence y. Furthermore, we have $x_m - y \in ces_p$ and also $x_m \in ces_p$

therefore $y = x_m - (x_m - y) \in ces_p$. Since (x_m) is an arbitrary Cauchy sequence in ces_p. Hence every Cauchy sequence is convergent in ces_p and ces_p is a Banach space.

Theorem 3.2 *Cesaro Sequence Space of an Absolut type ces_∞ equipped with norm function*

$$||X||_{ces_\infty} = \sup_{n\in\mathbb{N}}\left\{\frac{1}{n}\sum_{k=1}^{n}|x_k|\right\}$$

is a Banach space.

Proof:

a. Suppose $X = (x_n) \in ces_\infty$. Since $|x_k| \geq 0$ for every $k \in \mathbb{N}$, then we have

$$0 \leq \frac{1}{n}\sum_{k=1}^{n}|x_k| \leq \sup\left\{\frac{1}{n}\sum_{k=1}^{n}|x_k|, n \in \mathbb{N}\right\} < \infty$$
$$\Leftrightarrow 0 \leq ||X||_{ces_\infty} < \infty$$

□

b. $(\Rightarrow)$ If $X = (0,0,0,\dots)$ then

$$||X||_{ces_\infty} = \sup\left\{\frac{1}{n}\sum_{k=1}^{n}|0|, n \in \mathbb{N}\right\} = \sup\{0, \forall\, n \in \mathbb{N}\} = 0$$

$(\Leftarrow)$ Conversely, assume that $x \neq \mathbf{0}$ and $||x||_{ces_\infty} = 0$, then there exist $k_1 \in \mathbb{N}$ such that x_{k1} is the first non-zero term in x. Therefore

$$\frac{1}{n}\sum_{k=1}^{n}|x_k| > 0$$

For every $n > k_1$. And so we have

$$\sup\left\{\frac{1}{n}\sum_{k=1}^{n}|x_k|, n \in \mathbb{N}\right\} > 0$$

which contradict the fact that $||x||_{ces_\infty} = 0$. Hence $||x||_{ces_\infty} = 0$ if and only if $x = \mathbf{0}$.

c. Given any real number α, then

$$||\alpha X||_{ces_\infty} = \sup\left\{\frac{1}{n}\sum_{k=1}^{n}|\alpha x_k|, n \in \mathbb{N}\right\}$$
$$= \sup\left\{\frac{1}{n}\sum_{k=1}^{n}|\alpha||x_k|, n \in \mathbb{N}\right\}$$
$$= \sup\left\{\frac{|\alpha|}{n}\sum_{k=1}^{n}|x_k|, n \in \mathbb{N}\right\}$$
$$= |\alpha| \sup\left\{\frac{1}{n}\sum_{k=1}^{n}|x_k|, n \in \mathbb{N}\right\}$$
$$= |\alpha|||X||_{ces_\infty}$$

d. According to triangle inequality, we have $|x_k + y_k| < |x_k| + |y_k|$ for any real numbers x_k and y_k. Hence for $X, Y \in ces_p$ we have

$$||X+Y||_{ces_\infty} = \sup\left\{\frac{1}{n}\sum_{k=1}^{n}|x_k + y_k|, n \in \mathbb{N}\right\}$$
$$\leq \sup\left\{\frac{1}{n}\sum_{k=1}^{n}|x_k| + \frac{1}{n}\sum_{k=1}^{n}|y_k|, n \in \mathbb{N}\right\}$$
$$= \sup\left\{\frac{1}{n}\sum_{k=1}^{n}|x_k|, n \in \mathbb{N}\right\} + \sup\left\{\frac{1}{n}\sum_{k=1}^{n}|y_k|, n \in \mathbb{N}\right\}$$
$$= ||X||_{ces_\infty} + ||Y||_{ces_\infty}$$

e. From a, b, c, and d we have that ces_∞ is a norm space. Given $X = (x_n)$ be a Cauchy sequence on ces_∞ where $x_n = \left(x_n^{(k)}\right)$. Then for every positive real number ε, there exist $N \in \mathbb{N}$ such that any natural numbers m and q greater than or equal to N, we have

$$||x_m - x_q||_{ces_\infty} = \sup\left\{\frac{1}{n}\sum_{k=1}^{n}\left|x_m^{(k)} - x_q^{(k)}\right|, n \in \mathbb{N}\right\} < \varepsilon$$

That means for every $n \in \mathbb{N}$, we get

$$\frac{1}{n}\sum_{k=1}^{n}\left|x_m^{(k)} - x_q^{(k)}\right| < \varepsilon$$

Consequently, the sequence $\left(x^{(n)}\right) = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \dots)$ is a Cauchy sequence for any natural number n. Remember that $\left(x^{(n)}\right)$ is a Cauchy sequence on $\mathbb{R}$. Since $\mathbb{R}$ is a Banach space, therefore (x_n) is convergent $\forall n \in \mathbb{N}$. Suppose that $x_q^{(n)} \to y_n$ as $q \to \infty$. Then we can construct $y = (y_1, y_2, y_3, \dots)$. As of $q \to \infty$, we have

$$\sup\left\{\frac{1}{n}\sum_{k=1}^{n}\left|x_m^{(k)} - x_q^{(k)}\right|, n \in \mathbb{N}\right\} < \varepsilon$$
$$\sup\left\{\frac{1}{n}\sum_{k=1}^{n}\left|x_m^{(k)} - y_k\right|, n \in \mathbb{N}\right\} < \varepsilon$$
$$||x_m - y||_{ces_\infty} < \varepsilon$$

We can conclude that x_m is convergent to the sequence y. Furthermore, we have $x_m - y \in ces_\infty$ and also $x_m \in ces_\infty$ therefore $y = x_m - (x_m - y) \in ces_\infty$. Since (x_m) is an arbitrary Cauchy sequence in ces_∞. Hence every Cauchy sequence is convergent in ces_∞ and ces_∞ is a Banach space.

Theorem 3.3 *Cesaro Sequence Space of an Absolute type (ces_p) is a solid (normal) sequence space for every real number p satisfying $1 \leq p < \infty$.*

Proof:

Given any sequence $x \in ces_p$ where p is any real number that satisfy $1 \leq p < \infty$. Suppose (a_n) is an arbitrary real sequence where $|a_n| \leq 1$. For every $n \in \mathbb{N}$ we can get

$$\frac{1}{n}\sum_{k=1}^{n}|a_k x_k| = \frac{1}{n}\sum_{k=1}^{n}|a_k||x_k| < \frac{1}{n}\sum_{k=1}^{n}|x_k|$$

Therefore

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|a_n x_k|\right)^p\right)^{\frac{1}{p}} < \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{\frac{1}{p}} < \infty$$

Consequently $(a_n x_n) \in ces_p$ and ces_p is a solid sequence space for any real number $p \in [1, \infty)$.

Theorem 3.4 *Cesaro Sequence Space of an Absolute type (ces_∞) is a solid (normal) sequence space.*

Proof:

Given any sequence $x \in ces_\infty$. Suppose (a_n) is an arbitrary real sequence where $|a_n| \leq 1$. For every $n \in \mathbb{N}$ we can get

$$\frac{1}{n}\sum_{k=1}^{n}|a_k x_k| = \frac{1}{n}\sum_{k=1}^{n}|a_k||x_k| < \frac{1}{n}\sum_{k=1}^{n}|x_k|$$

Therefore

$$\sup_{n\in\mathbb{N}}\left\{\frac{1}{n}\sum_{k=1}^{n}|a_k x_k|\right\} < \sup_{n\in\mathbb{N}}\left\{\frac{1}{n}\sum_{k=1}^{n}|x_k|\right\} < \infty$$

Hence $(a_n x_n) \in ces_\infty$ and ces_∞ is a solid sequence space.

Theorem 3.5 *Cesaro Sequence Space of an Absolute type (ces_p) is a BK-space for every real number p satisfying $1 \leq p < \infty$.*

Proof:

Given an arbitrary natural number N and $x \in ces_p$. Then for every real number $\varepsilon > 0$, there exist positive real number $\delta = \frac{\varepsilon}{N}$ such that for every $y \in ces_p$ where $||y - x||_{ces_p} < \delta$, we have

$$||y - x||_{ces_p} = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|y_k - x_k|\right)^p\right)^{\frac{1}{p}} < \delta$$

$$\Leftrightarrow \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|y_k - x_k|\right)^p < \delta^p$$

That means for $n = N$ we get

$$\left(\frac{1}{N}\sum_{k=1}^{N}|y_k - x_k|\right)^p < \delta^p$$

$$\frac{1}{N}\sum_{k=1}^{N}|y_k - x_k| < \delta$$

$$|y_N - x_N| \leq \sum_{k=1}^{N}|y_k - x_k| < N\delta$$

$$|p_N(y) - p_N(x)| < N\left(\frac{\varepsilon}{N}\right) = \varepsilon$$

Hence the function $p_N: ces_p \to \mathbb{C}$ where $p_N(x) = x_N$ is continuous for every $N \in \mathbb{N}$. Since ces_p is a Banach space (Theorem 3.1), then ces_p is a BK-space. The proof is complete.

Theorem 3.6 *Cesaro Sequence Space of an Absolute type (ces_∞) is a BK-space.*

Proof:

Given an arbitrary natural number N and $x \in ces_\infty$. Then for every real number $\varepsilon > 0$, there exist positive real number $\delta = \frac{\varepsilon}{N}$ such that for every $y \in ces_\infty$ where $||y - x||_{ces_\infty} < \delta$, we have

$$||y - x||_{ces_\infty} = \sup_{n\in\mathbb{N}}\left\{\frac{1}{n}\sum_{k=1}^{n}|y_k - x_k|\right\} < \delta$$

That means for $n = N$ we get

$$\frac{1}{N}\sum_{k=1}^{N}|y_k - x_k| < \delta$$

$$|y_N - x_N| \leq \sum_{k=1}^{N}|y_k - x_k| < N\delta$$

$$|p_N(y) - p_N(x)| < N\left(\frac{\varepsilon}{N}\right) = \varepsilon$$

Hence the function $p_N: ces_\infty \to \mathbb{C}$ where $p_N(x) = x_N$ is continuous for every $N \in \mathbb{N}$. Since ces_∞ is a Banach space (Theorem 3.2), then ces_∞ is a BK-space. The proof is complete.

Corollary 3.7 *ces_p is an FK-space whenever p is a real number satisfying $1 \leq p < \infty$.*

Proof:

The proof is directly from Theorem 3.5 and Lemma 2.9.

Corollary 3.8 *ces_∞ is an FK-space.*

Proof:

The proof is directly from Theorem 3.6 and Lemma 2.9.

Theorem 3.9 *Cesaro Sequence Space of an Absolute type (ces_p) have the AK-property for every real number p satisfying $1 \leq p < \infty$.*

Proof:

From Corollary 3.7 we know that ces_p is an FK-space. Given an arbitrary $x = (x_n) \in ces_p$. Then we have

$$\left|\left|\sum_{k=1}^{r}x_k e^{(k)} - x\right|\right|_{ces_p}^p = \sum_{n=r+1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p$$

$$= ||x||^p_{ces_p} - \sum_{n=1}^{r}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p$$

Therefore we can construct a sequence $(a_r) = \left(b - \sum_{n=1}^{r}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)$ with $b = ||x||^p_{ces_p}$. We can see that $\forall r \in \mathbb{N}$ we have $0 \leq a_r \leq b$. Also

$$a_r = b - \sum_{n=1}^{r}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p > b - \sum_{n=1}^{r+1}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p = a_{r+1}$$

Therefore (a_r) is bounded and decreasing. Hence it is convergent to its infimum which is 0. Consequently, as $r \to \infty$ we have

$$\left|\left|\sum_{k=1}^{r}x_k e^{(k)} - x\right|\right|_{ces_p}^p = 0 \Leftrightarrow \sum_{k=1}^{r}x_k e^{(k)} = x$$

Hence ces_p have the AK-property.

Theorem 3.10 *Cesaro Sequence Space of an Absolute type (ces_∞) have the AK-property.*

Proof:

From Corollary 3.8 we know that ces_∞ is an FK-space. Given an arbitrary $x = (x_n) \in ces_\infty$. Then we have

$$\left|\left|\sum_{k=1}^{r}x_k e^{(k)} - x\right|\right|_{ces_\infty} = \sup_{n\in\mathbb{N}}\left\{\frac{1}{n}\sum_{k=r+1}^{n}|x_k|\right\}$$

$$= \sup_{n\in\mathbb{N}}\left\{\frac{1}{n}\sum_{k=1}^{n}|x_k| - \frac{1}{n}\sum_{k=1}^{r}|x_k|\right\}$$

Where for $r \geq n$ we define $\frac{1}{n}\sum_{k=1}^{n}|x_k| - \frac{1}{n}\sum_{k=1}^{r}|x_k| = 0$. Therefore, we can construct a sequence $(a_r) =$

$\left(\sup_{n\in\mathbb{N}}\left\{\frac{1}{n}\sum_{k=1}^{n}|x_k| - \frac{1}{n}\sum_{k=1}^{r}|x_k|\right\}\right)$. We can see that $\forall r \in \mathbb{N}$ we have

$$0 < \frac{1}{n}\sum_{k=1}^{n}|x_k| - \frac{1}{n}\sum_{k=1}^{r}|x_k| < \frac{1}{n}\sum_{k=1}^{n}|x_k|$$
$$\Leftrightarrow 0 \le \sup_{n\in\mathbb{N}}\left\{\frac{1}{n}\sum_{k=1}^{n}|x_k| - \frac{1}{n}\sum_{k=1}^{r}|x_k|\right\} < ||x||_{ces_\infty}$$
$$\Leftrightarrow 0 \le a_r \le ||x||_{ces_\infty}$$

And also

$$a_r = \sup_{n\in\mathbb{N}}\left\{\frac{1}{n}\sum_{k=1}^{n}|x_k| - \frac{1}{n}\sum_{k=1}^{r}|x_k|\right\} > \sup_{n\in\mathbb{N}}\left\{\frac{1}{n}\sum_{k=1}^{n}|x_k| - \frac{1}{n}\sum_{k=1}^{r+1}|x_k|\right\} = a_{r+1}$$

Therefore (a_r) is bounded and decreasing. Hence it is convergent to its infimum which is 0. Consequently, as $r \to \infty$ we have

$$\left\|\left\|\sum_{k=1}^{r} x_k e^{(k)} - x\right\|\right\|_{ces_\infty}^{p} = 0 \Leftrightarrow \sum_{k=1}^{r} x_k e^{(k)} = x$$

Hence ces_∞ have the AK-property.

4. Conclusion

In this article, we can conclude that Cesaro Sequence Space of an Absolute type is a Banach space when equipped with norm $||X||_{ces_p} = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{\frac{1}{p}}$ for any real numbers p satisfying $1 \le p < \infty$ and $||X||_{ces_\infty} = \sup_{n\in\mathbb{N}}\left\{\frac{1}{n}\sum_{k=1}^{n}|x_k|\right\}$ when $p = \infty$. Furthermore, both ces_p and ces_∞ are also a solid sequence space, a BK-space, an FK-space, and having AK-property.

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