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Cassini Oval to Limacon: An Analytic Conversion

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***Abstract*: This paper illustrates how a Cassini Oval can be converted to a Limacon using analytic Geometry.**

***Keywords*: Cassini Oval, Limacon, Quartic plane curves, Analytic conversion.**

1. Introduction

A *Cassini Oval* is a quartic plane curve defined as the locus of a point in the plane such that the product of the distances of the point from two fixed points is a constant. The fixed points are called foci. The curve was first investigated by *Giovanni Cassini* in 1680 when he was studying the relative motions of the Earth and the Sun.

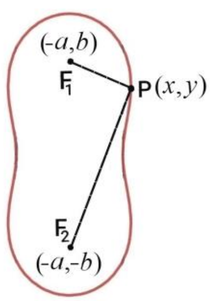

Fig. 1

A *Cassini Oval* [Fig.1] with foci at $(-a, b)$, $(-a,-b)$ and the constant product of the distances k^2 can be represented by the Cartesian equation

$$\left((x+a)^2+(y-b)^2\right)\left((x+a)^2+(y+b)^2\right)=k^4 \qquad (1.1)$$

When expanded (1.1) becomes

$$(x^2+y^2)^2+4ax(x^2+y^2)+2(3a^2+b^2)x^2+2(a^2-b^2)y^2$$
$$+4a(a^2+b^2)x+\left((a^2+b^2)^2-k^4\right)=0 \qquad (1.2)$$

On the other hand, a *Limacon* is a quartic plane curve defined as the locus of a point fixed to a circle when the circle rolls around the outside of a circle of equal radius. The earliest formal research on Limacon is generally attributed to *Etienne Pascal*, father of *Blaise Pascal*. However the curve was later renamed by *Gilles Personne Roberval* in 1650 when he used it as an example for finding tangent lines.

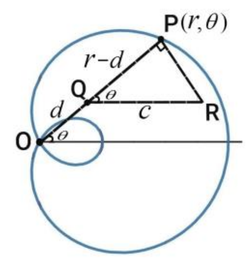

Fig. 2

A *Limacon* [Fig.2] is usually expressed in Polar coordinates as $r = d + c\cos\theta$. The same can be represented by the Cartesian equation

$$(x^2+y^2-cx)^2 = d^2(x^2+y^2) \qquad (1.3)$$

When expanded (1.3) becomes

$$(x^2+y^2)^2-2cx(x^2+y^2)+(c^2-d^2)x^2-d^2y^2=0 \qquad (1.4)$$

Shifting the origin at $(-\lambda, 0)$, equation (1.4) changes to

$$\left((x+\lambda)^2+y^2\right)^2-2c(x+\lambda)\left((x+\lambda)^2+y^2\right)$$
$$+(c^2-d^2)(x+\lambda)^2-d^2y^2=0 \qquad (1.5)$$

When expanded (1.5) becomes

$$(x^2+y^2)^2+2(2\lambda-c)x(x^2+y^2)+\left((c^2-d^2)+6\lambda(\lambda-c)\right)x^2$$
$$+\left(2\lambda(\lambda-c)-d^2\right)y^2+2\lambda\left((c^2-d^2)+\lambda(2\lambda-3c)\right)x$$
$$+\lambda^2\left((\lambda-c)^2+d^2\right)=0 \qquad (1.6)$$

Our target is to develop a technique that would enable us to convert a *Cassini Oval* to a *Limacon*. For this purpose, we are going to use a quadratic function

$$f(z) = Az^2 + Bz + C$$

; where *A*, *B*, *C* are real and $A \neq 0$.

First we are going to convert a *Cassini Oval* to the unit circle and then the unit circle will be finally converted to a *Limacon*.

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2. A Cassini Oval to the Unit Circle

Let $f(z) = Az^2 + Bz + C$ lie on the unit circle $x^2 + y^2 = 1$.

$$\Rightarrow \left| f(z) \right| = 1$$

$$\Rightarrow \left| Az^2 + Bz + C \right| = 1$$

$$\Rightarrow \left| Az^2 + Bz + C \right|^2 = 1$$

Let $z = x + iy$; $x, y \in R$, $i = \sqrt{-1}$.

$$\Rightarrow \left| A(x+iy)^2 + B(x+iy) + C \right|^2 = 1$$

$$\Rightarrow \left(A(x^2 - y^2) + (Bx + C)\right)^2 + \left(y\,(2Ax + B)\right)^2 = 1$$

When expanded this becomes

$$A^2(x^2+y^2)^2 + 2ABx(x^2+y^2) + (B^2+2AC)x^2 + (B^2-2AC)y^2 + 2BCx + (C^2-1) = 0 \quad (2.1)$$

Comparing the coefficients of (1.2) and (2.1), we get

$$\frac{1}{A^2} = \frac{4a}{2AB} = \frac{2(3a^2+b^2)}{B^2+2AC} = \frac{2(a^2-b^2)}{B^2-2AC} = \frac{4a(a^2+b^2)}{2BC} = \frac{(a^2+b^2)^2 - k^4}{C^2-1} \quad (2.2)$$

On simplification (2.2) yields

$$a = \frac{B}{2A} \quad (2.3)$$

$$b = \frac{\pm\sqrt{4AC - B^2}}{2A} \quad (2.4)$$

$$(a^2+b^2)^2 - k^4 = \frac{C^2-1}{A^2} \quad (2.5)$$

Squaring and adding (2.3) and (2.4), we get

$$a^2 + b^2 = \frac{C}{A} \quad (2.6)$$

Combining (2.5) and (2.6), further we get

$$\frac{C^2}{A^2} - k^4 = \frac{C^2-1}{A^2} \quad (2.7)$$

On simplification (2.7) yields

$$A = \frac{\pm 1}{k^2} \quad (2.8)$$

So only two cases are possible.

Case 1 $\quad A = \frac{1}{k^2},\ B = \frac{2a}{k^2}, C = \frac{(a^2+b^2)}{k^2} \quad (2.9)$

Case 2 $\quad A = \frac{-1}{k^2},\ B = \frac{-2a}{k^2},\ C = \frac{-(a^2+b^2)}{k^2} \quad (2.10)$

As $\left| Az^2 + Bz + C \right| = 1$ and $\left| (-A)z^2 + (-B)z + (-C) \right| = 1$ are same, hence we are going to consider the values of *A*, *B* and *C* as obtained in (2.9). However the values of *A*, *B* and *C* as obtained in (2.10) may also be considered.

Again equation (2.1), dividing each side by A^2, can be rewritten as

$$(x^2+y^2)^2 + 2\left(\frac{B}{A}\right)x(x^2+y^2) + \left(\frac{B^2+2AC}{A^2}\right)x^2 + \left(\frac{B^2-2AC}{A^2}\right)y^2 + 2\left(\frac{BC}{A^2}\right)x + \left(\frac{C^2-1}{A^2}\right) = 0 \quad (2.11)$$

Thus, the locus of z is

$$\left\{ z = x+iy : (x^2+y^2)^2 + 2\left(\frac{B}{A}\right)x(x^2+y^2)^2 + \left(\frac{B^2+2AC}{A^2}\right)x^2 + \left(\frac{B^2-2AC}{A^2}\right)y^2 + 2\left(\frac{BC}{A^2}\right)x + \left(\frac{C^2-1}{A^2}\right) = 0 \right\} \quad (2.12)$$

So the set points describing *z* is a *Cassini Oval*. The size, shape and location of this curve are subject to change depending upon various real values of *A*, *B* and *C*. However the curve is merely translated (along the real axis) without any alteration in size and shape if any change is incorporated in *C*, subject to the condition both *A* and *B* remain unchanged. In particular, when $a = 1/2$, $b = \sqrt{3}/2$ and $k^2 = 1$; then *A* =1, *B* =1 and *C* =1. Then the solution set of z is given by

$$\left\{ z = x+iy : (x^2+y^2)^2 + 2\,x(x^2+y^2)^2 + 3x^2 - y^2 + 2\,x = 0 \right\} \quad (2.13)$$

Let's plot both *z* as well as *f* (*z*) on the Cartesian plane to understand the geometry of conversion. When *z* is on the *Cassini Oval*, $f(z)$ is on the unit circle, [Fig. 3].

In other words, existence of *z* on the *Cassini Oval* implies existence of $f(z)$ on the unit circle. So f is the analytic function that converts a *Cassini Oval* to the unit circle.

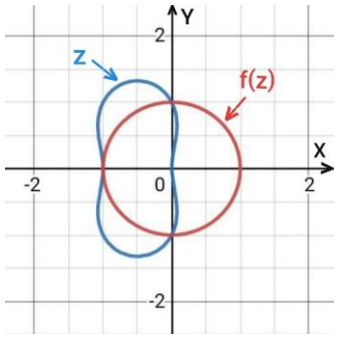

Fig. 3

3. The Unit Circle to a Limacon

Let z lie on the unit circle $x^2+y^2=1$.

$\Rightarrow |z|=1$

So we take $z=e^{i\theta}$.

$\therefore f(z)=Az^2+Bz+C$

$=Ae^{i\theta}+Be^{i\theta}+C$

$=A(\cos 2\theta+i\sin 2\theta)+B(\cos\theta+i\sin\theta)+C$

Suppose $f(z)=x+i\,y$; $x,y\in R$, $i=\sqrt{-1}$.

Then $x+i\,y=(A\cos 2\theta+B\cos\theta+C)+i(A\sin 2\theta+B\sin\theta)$.

Separating real and imaginary parts, we get

$$x-C=A\cos 2\theta+B\cos\theta \quad (3.1)$$

$$y=A\sin 2\theta+B\sin\theta \quad (3.2)$$

Squaring and adding (3.1) and (3.2), further we get

$$(x-C)^2+y^2=A^2+B^2+2AB\cos\theta \quad (3.3)$$

$$\Rightarrow \cos\theta=\frac{(x-C)^2+y^2-(A^2+B^2)}{2AB} \quad (3.4)$$

Again (3.1) can be rewritten as

$$x-C=A(2\cos^2\theta-1)+B\cos\theta \quad (3.5)$$

$$\Rightarrow x-C=2A\left(\frac{(x-C)^2+y^2-(A^2+B^2)}{2AB}\right)^2 +B\left(\frac{(x-C)^2+y^2-(A^2+B^2)}{2AB}\right)-A \quad (3.6)$$

When expanded this becomes

$$(x^2+y^2)^2-4Cx(x^2+y^2) +(6C^2-B^2-2A^2)x^2+(2C^2-B^2-2A^2)y^2 +2(A-C)(2C^2-B^2+2AC)x +(A-C)^2(A+B+C)(A-B+C)=0 \quad (3.7)$$

Thus, the locus of $f(z)$ is

$$\{\, f(z)=x+iy:(x^2+y^2)^2-4Cx(x^2+y^2) +(6C^2-B^2-2A^2)x^2+(2C^2-B^2-2A^2)y^2 +2(A-C)(2C^2-B^2+2AC)x +(A-C)^2(A+B+C)(A-B+C)=0 \,\} \quad (3.8)$$

So the set points describing *z* is a *Limacon*. The size, shape and location of this curve are subject to change depending upon various real values of *A*, *B* and *C*. However the curve is merely translated (along the real axis) without any alteration in size and shape if any change is incorporated in *C*, subject to the condition both *A* and *B* remain unchanged. In particular, when when $A=1$, $B=1$ and $C=1$; then the solution set of $f(z)$ is given by

$$\{\, f(z)=x+iy:(x^2+y^2)^2-4x(x^2+y^2) +3x^2-y^2=0 \,\} \quad (3.9)$$

Let's plot both *z* as well as *f* (*z*) on the Cartesian plane to understand the geometry of conversion. When *z* is on the unit circle, *f* (*z*) is on the *Limacon* [Fig. 4].

In other words, existence of *z* on the unit circle implies existence of *f* (*z*) on the *Limacon*. So *f* is the analytic function that converts the unit circle to a *Limacon*.

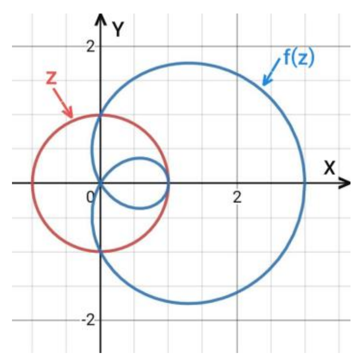

Fig. 4

4. A Cassini Oval to a Limacon

Finally, we apply both the transformations successively; a *Cassini Oval* to the unit circle followed by the unit circle to a *Limacon*. To achieve the goal, we consider a point *z* on the *Cassini Oval*. It returns us a point *f* (*z*) back on the unit circle. Then, we get another point *f* (*f* (*z*)) on the *Limacon*. In this way, the composite analytic function *fof* (*z*) enables us to convert a *Cassini Oval* to a *Limacon* [Fig. 5].

Now let's discuss the necessary steps involved in this process of conversion.

Step 1 Get the equation of the *Cassini oval* by suitably placing the coordinate axes such that the foci are at $(-a, b)$ and $(-a,-b)$.

Step 2 Note down the value of k^2 (product of the distances of any point on the *Cassini Oval*) and also the values of a and b.

Step 3 Calculate the values of *A*, *B* and *C* using the following formulae

$$A = \frac{1}{k^2},\ B = \frac{2a}{k^2},\quad C = \frac{(a^2+b^2)}{k^2}.$$

Step 4 Transform every point *z* on the *Cassini oval* to the corresponding point on the *Limacon* using the composite transformation function $fof(z)$; where $f(z) = Az^2 + Bz + C$.

$$\therefore\ fof(z) = A((f(z))^2 + B(f(z)) + C$$

When expanded this becomes

$$fof(z) = A^3z^4 + 2A^2Bz^3 + (AB^2 + 2A^2C + AB)z^2 + (2ABC + B^2)z + (AC^2 + BC + C) \quad (4.1)$$

In particular, when *A*=1, *B*=1 and *C*=1; the function given by (4.1) reduces to

$$fof(z) = z^4 + 2z^3 + 4z^2 + 3z + 3 \quad (4.2)$$

The analytic function (4.2) transforms every point *z* on the *Cassini Oval* given by (2.13) to the corresponding point on the *Limacon* given by (3.9).

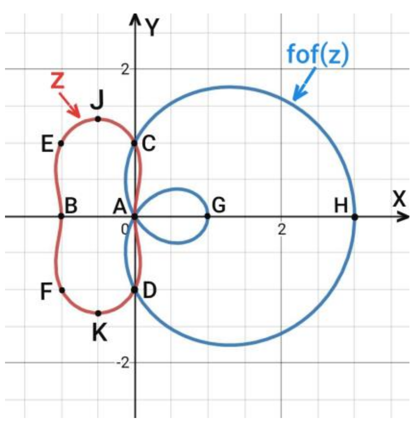

Fig. 5

As an experiment [Table 1 and Table 2], just to check the correctness of the conversion numerically, one can take few points *z* on the *Cassini Oval* as inputs and investigate the outputs $fof(z)$ lie on what. For example the points *A*, *B*, *C*, *D*, *E*, *F*, *J* and *K* are the inputs and the corresponding outputs are *H*, *H*, *C*, *D*, *D*, *C*, *G* and *G* respectively. That's pretty good, the outputs are on the *Limacon* [Fig. 5].

However, when $A = -1$, $B = -1$ and $C = -1$; the function given by (4.1) reduces to

$$fof(z) = -z^4 - 2z^3 - 2z^2 - z - 1 \quad (4.3)$$

Then the solution set of $f(z)$ is given by

$$\{\ f(z) = x + iy : (x^2 + y^2)^2 + 4x(x^2 + y^2) + 3x^2 - y^2 = 0\ \} \quad (4.4)$$

The *Limacon* given by (3.9) and the *Limacon* given by (4.4) are equal in size but mirror image to each other with respect to the imaginary axis.

Further instead of $fof(z) = z^4 + 2z^3 + 4z^2 + 3z + 3$ if $fof(z) = z^4 + 2z^3 + 4z^2 + 3z$ is chosen as the analytic function for the purpose of conversion, an equal sized *Limacon* would be obtained which is translated 3 units along the negative direction of the real axis.

Similarly in lieu of $fof(z) = -z^4 - 2z^3 - 2z^2 - z - 1$ if $fof(z) = -z^4 - 2z^3 - 2z^2 - z$ is chosen as the analytic function for the purpose of conversion, an equal sized *Limacon* would be obtained which is translated 1 unit along the positive direction of the real axis.

Table 1

z	*A*(0)	*B*(-1)	*C*(*i*)	*D*(-*i*)	*E*(-1+*i*)	*F*(-1-*i*)
fof (z)	*H*(3)	*H*(3)	*C*(*i*)	*D*(-*i*)	*D*(-*i*)	*C*(*i*)

Table 2

z	$J\left(\frac{-1}{2}+\frac{\sqrt{7}}{2}i\right)$	$K\left(\frac{-1}{2}-\frac{\sqrt{7}}{2}i\right)$
fof (z)	*G*(1)	*G*(1)

References

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[2] Limacon of Pascal, MacTutor History of Mathematics.

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